3.1 Calculus

There are a couple of great webpages for checking your answers to Calculus problems...

Check Differentiation answers
Check Integration answers

3.1.1 The Concept of the Limit, Continuity and the Derivative

Calculus deals with functions which continually vary and is based upon the concept of a limit and continuity. Let’s refer to a diagram to understand the concept of a limit...

Here we have indicated a point P on part of a function (the red curve) with Cartesian co-ordinates \( (x_1, y_1) \). We require another point Q on the function to be a small increment away from P and will designate a small increment with the symbol \( \delta \) (delta).

What we then have is...

\[ P = (x_1, y_1) \]
\[ Q = (x_1 + \delta x, y_1 + \delta y) \]

Look now at the chord PQ (drawn as the straight line in blue). If we can determine the slope of this chord and then make it infinitesimally short we will end up with a tangent to the function. It is the slope of this tangent which forms the basis of Differential Calculus (normally called differentiation).

By inspection, we see that the slope of the chord is given by...

\[ \text{slope of chord } PQ = \frac{\delta y}{\delta x} \]

If we deliberately make \( \delta x \) approach zero (i.e. make it as short as possible) then we shall reach a limit, which may expressed mathematically as...
\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}
\]

The term \(dy/dx\) is written in Leibnitz notation and indicates the slope of the chord when the chord only touches the function at one single point. This is achieved by continual reduction of \(\delta x\).

What we can now say is that we are able to find the slope of any function by adopting this process. Hence, given a function \(f(x)\) we are able to differentiate that function, meaning find its slope at all points. We can write...

The slope at any point of a function \(f(x)\) is given by \(\frac{d(f(x))}{dx}\)

*This process is called finding the derivative of a function.*

### 3.1.2 Derivatives of Standard Functions

What we don’t want to be doing is to spend too much time drawing graphs of functions just to work out the derivative. Fortunately there are standard ways to determine the derivative of functions and some of the frequent ones which engineers meet are given in the table below.

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Ax^n)</td>
<td>(nAx^{n-1})</td>
</tr>
<tr>
<td>(A \sin(x))</td>
<td>(A \cos(x))</td>
</tr>
<tr>
<td>(A \cos(x))</td>
<td>(-A \sin(x))</td>
</tr>
<tr>
<td>(A e^{kx})</td>
<td>(kAe^{kx})</td>
</tr>
<tr>
<td>(A \log_e(x))</td>
<td>(\frac{A}{x})</td>
</tr>
<tr>
<td>(A \sinh(x))</td>
<td>(A \cosh(x))</td>
</tr>
<tr>
<td>(A \cosh(x))</td>
<td>(A \sinh(x))</td>
</tr>
</tbody>
</table>

Let’s look at some examples of using these standard derivatives...

**Worked Example 1**

**Differentiate the function** \(y = 3x^4\) **with respect to** \(x\).

We see that this function is similar to that in row 1 of our table. We can see that \(A = 3\) and \(n = 4\). We may then write...
\[
\frac{dy}{dx} = nAx^{n-1} = (4)(3)x^{4-1} = 12x^3
\]

**Worked Example 2**

Differentiate the function \( y = 6x^{-5} \) with respect to \( x \).

We see that this function is similar to that in row 1 of our table. We can see that \( A = 6 \) and \( n = -5 \). We may then write...

\[
\frac{dy}{dx} = nAx^{n-1} = (-5)(6)x^{-5-1} = -30x^{-6}
\]

**Worked Example 3**

Differentiate the function \( v = 12 \sin(t) \) with respect to \( t \).

Don’t worry that \( y \) and \( x \) have disappeared here. We can use any letters we like. The letter \( v \) means voltage and \( t \) is time. All we want to do here then is to find \( \frac{dv}{dt} \).

We see that the function is similar to that in row 2 of our table. We may write...

\[
\frac{dv}{dt} = A \cos(t) = 12 \cos(t)
\]

**Worked Example 4**

Differentiate the function \( i = 5 \cos(t) \) with respect to \( t \).

The letter \( i \) means current and \( t \) is time. All we want to do here then is to find \( \frac{di}{dt} \).

We see that the function is similar to that in row 3 of our table. We may write...

\[
\frac{di}{dt} = -A \sin(t) = -5 \sin(t)
\]
Worked Example 5

Differentiate the function \( y = 4e^{2x} \) with respect to \( x \).

From row 4 of our table we can write...

\[
\frac{dy}{dx} = kAe^{kx} = (2)(4)e^{2x} = 8e^{2x}
\]

Worked Example 6

Differentiate the function \( y = 16 \log_e(x) \) with respect to \( x \).

We refer to row 5 of our table and write...

\[
\frac{dy}{dx} = \frac{A}{x} = \frac{16}{x}
\]

Worked Example 7

Differentiate the function \( y = 9 \sinh(x) \) with respect to \( x \).

We refer to row 6 of our table and write...

\[
\frac{dy}{dx} = A \cosh(x) = 9 \cosh(x)
\]

Worked Example 8

Differentiate the function \( y = -14.5 \cosh(x) \) with respect to \( x \).

We refer to row 7 of our table and write...

\[
\frac{dy}{dx} = A \sinh(x) = -14.5 \sinh(x)
\]

Questions

Q3.1 Differentiate the function \( y = 8x^5 \) with respect to \( x \).
Q3.2 Differentiate the function $y = 7x^{-4}$ with respect to $x$.

Q3.3 Differentiate the function $v = 18 \sin(t)$ with respect to $t$.

Q3.4 Differentiate the function $i = 11 \cos(t)$ with respect to $t$.

Q3.5 Differentiate the function $y = 6e^{3x}$ with respect to $x$.

Q3.6 Differentiate the function $y = 20 \log_e(x)$ with respect to $x$.

Q3.7 Differentiate the function $y = 13 \sinh(x)$ with respect to $x$.

Q3.8 Differentiate the function $y = -19.62 \cosh(x)$ with respect to $x$.

ANSWERS

| Q3.1 | 40x^4 |
| Q3.2 | -28x^{-5} |
| Q3.3 | 18 \cos(t) |
| Q3.4 | -11 \sin(t) |
| Q3.5 | 18e^{3x} |
| Q3.6 | 20/x |
| Q3.7 | 13 \cosh(x) |
| Q3.8 | -19.62 \sinh(x) |

These videos will boost your knowledge of differentiation

3.1.3 Notion of the Derivative and Rates of Change
The notion of taking the derivative of a function and examining rate of change can be readily explained with reference to the diagram below...
Here we have plotted a section of a sine wave in red. Notice that we have used pink arrows to indicate the slope at various points on the sine wave. For example, at the start the sine wave is rising to a positive more rapidly than at any other point. The pink arrow indicates this positive growth by pointing upwards in the direction of the sine wave at that beginning point. The second arrow shows the slope of the sine wave at $\pi/2$ radians (or $90^\circ$ if you like). Here it is at a plateau and has zero slope (meaning that it just moves horizontally, with no vertical movement at all). The third arrow indicates a negative slope (its moving down very steeply). The picture continues in a similar way for the remaining arrows.

Here’s the really interesting bit. When we record all of the slopes on our sine wave we get the dotted waveform in blue. What does that look like? Of course, it is a cosine wave. So we have just proved graphically that finding the slope (differentiating) at all points on a sine wave leads to a cosine wave.

$$\frac{d}{dt}\{\sin(t)\} = \cos(t)$$

... as given in our table of standard derivatives in section 3.1.2.

If we had loads of time to kill we could prove all of the other derivatives graphically.

**Challenge**

Determine a graphical proof (as above) for...

$$\frac{d}{dt}\{\cos(t)\} = -\sin(t)$$
3.2.3 Differentiation of Inverse Trigonometric Functions

Suppose we have the equation...

$$y = \sin^{-1}(x)$$

How do we then go about finding \(dy/dx\)? Maybe we should write this in a different form...

$$x = \sin(y)$$

Now can differentiate...

$$\frac{dx}{dy} = \cos(y) \quad \text{flip both sides:} \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Next we shall remember one of our handy trig. Identities...

$$\cos^2(y) + \sin^2(y) = 1 \quad \therefore \quad \cos^2(y) = 1 - \sin^2(y) \quad \therefore \quad \cos(y) = \sqrt{1 - \sin^2(y)}$$

Since we know \(x = \sin(y)\) then if we square both sides we shall have \(x^2 = \sin^2(y)\)

So we may re-write our transposed trig. identity as...

$$\cos(y) = \sqrt{1 - x^2}$$

Now we can put this latest result back into our differential...

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{(1 - x^2)^{0.5}} = (1 - x^2)^{-0.5}$$

That was a bit tricky, but we managed to find \(y'\).

Let’s go on and see if we can find \(y''\)...

$$y' = (1 - x^2)^{-0.5}$$

This is where that shorthand notation for derivatives comes in really handy...

Let \(u = 1 - x^2\) \quad \therefore \quad \frac{du}{dx} = -2x

\[\therefore \quad y' = u^{-0.5}\]

\[\therefore \quad \frac{dy'}{du} = -0.5u^{-1.5}\]

\[\therefore \quad y'' = \frac{du}{dx} \cdot \frac{dy'}{du} = -2x \times -0.5u^{-1.5}\]

\[\therefore \quad y'' = xu^{-1.5} = x(1 - x^2)^{-1.5}\]
Q3.23 Find $y''$ for $y = \cos^{-1}(2x)$

ANSWER

\[
Q3.23 \quad \frac{-8x}{(1-4x^2)^{1.5}}
\]

3.2.4 Differentiation of Inverse Hyperbolic Functions

Consider the function...

\[ y = \sinh^{-1}(x) \]

Our task here is to find $y''$. We proceed in a similar manner to the previous problem...

\[ x = \sinh(y) \]

\[ \therefore \frac{dx}{dy} = \cosh(y) \quad \therefore \frac{dy}{dx} = \frac{1}{\cosh(y)} \]

A useful identity is...

\[ \cosh^2(y) - \sinh^2(y) = 1 \quad \therefore \cosh^2(y) = 1 + \sinh^2(y) \]

\[ \therefore \cosh^2(y) = 1 + x^2 \quad \therefore \cosh(y) = \sqrt{1 + x^2} \]

\[ \therefore \frac{dy}{dx} = y' = \frac{1}{\sqrt{1 + x^2}} = (1 + x^2)^{-0.5} \]

Now to find that second derivative...

\[ y' = (1 + x^2)^{-0.5} \]

Let $u = 1 + x^2$ \: \therefore \frac{du}{dx} = 2x

\[ \therefore y' = u^{-0.5} \quad \therefore \frac{dy'}{du} = -0.5u^{-1.5} \]

\[ \therefore y'' = \frac{du}{dx} \cdot \frac{dy'}{du} = 2x \times -0.5u^{-1.5} = -x(1 + x^2)^{-1.5} \]

Q3.24 Find $y''$ for $y = \cosh^{-1}(2x)$

ANSWER
3.3 Further Integration

3.3.1 Integration by Parts
This technique provides us with a way to integrate the product of two simple functions. We shall develop a formula to use as a framework for the technique.

The starting point for our formula lies with the Product Rule, which you mastered in Section 3.1.5...

\[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

Let’s rearrange this formula...

\[ u \frac{dv}{dx} = \frac{d}{dx}(uv) - v \frac{du}{dx} \]

If we now integrate both sides with respect to \( x \) then we need to place a \( \int \) sign before each term and a \( dx \) at the end of each term, like so...

\[ \int u \frac{dv}{dx} \, dx = \int \frac{d}{dx}(uv) \, dx - \int v \frac{du}{dx} \, dx \]

Some simplifications can be noticed here:

- In the first integral the two \( dx \) terms cancel each other out
- In the second integral we are taking the ‘integral of the differential’ of \( uv \). Since integration is the reverse of differentiation then the ‘integral of a differential’ drops away, just leaving \( uv \)
- In the third integral the two \( dx \) terms cancel each other out

Performing these simplifications gives...

\[ \int u \, dv = uv - \int v \, du \]

That is our formula for integration by parts.

What we do with the formula is to look at the left hand side and see the integral of a product. That product is composed of \( u \) and \( dv \). When using integration by parts you have a choice of which part of the product to call \( u \) and which to call \( dv \). Fortunately, your choice is guided...

- The \( u \) part will become a constant after taking multiple derivatives
- The \( dv \) part is readily integrated by using standard integrals.
Determine: $\int x \sin(x) \, dx$

One part of our product is $x$ and the other part is $\sin(x)$. We need to make a choice as to which one to make $u$ and which to make $dv$. Our guidance tells us to make the $x$ part equal to $u$ since the derivative of $x$ becomes 1 (i.e. a constant)...

Let $u = x$

Let $dv = \sin(x)$

We now try to form the terms in our integration by parts formula...

If $u = x \quad \therefore \quad \frac{du}{dx} = 1 \quad \therefore \quad du = dx$

If $dv = \sin(x) \quad \therefore \quad v = -\cos(x)$

We can now write...

$$\int u \, dv = uv - \int v \, du$$

$$\int x \sin(x) \, dx = x(-\cos(x)) - \int (-\cos(x)) \, dx$$

$$\therefore \quad \int x \sin(x) \, dx = -x \cos(x) + \int \cos(x) \, dx$$

$$\therefore \quad \int x \sin(x) \, dx = -x \cos(x) + \sin(x) + C$$

That was fairly straightforward. Always remember the constant of integration $C$ at the end.

Let's now try to integrate the product of an exponential and a cosine.

Determine: $\int e^{2x} \cos(x) \, dx$

Our two products here are $e^{2x}$ and $\cos(x)$. Neither of them will reduce to a constant when differentiated so it doesn’t matter which one we call $u$ and which we call $dv$. You will see here that we need to do integration by parts twice to arrive at our answer.

Just as in the previous worked example we need to decide on assignments for $u$ and $dv$, then find $v$ and $du$. Let’s do that...

Let $u = e^{2x} \quad \therefore \quad \frac{du}{dx} = 2e^{2x} \quad \therefore \quad du = 2e^{2x} \, dx$
Let \( dv = \cos(x) \) \( \therefore v = \sin(x) \)

To aid our working here (and save some ink) we normally assign the capital letter \( I \) to the integral in our question.

\[
\therefore I = \int e^{2x} \cos(x) \, dx
\]

\[
\therefore I = e^{2x} \sin(x) - \int \sin(x) 2e^{2x} \, dx
\]

This needs tidying...

\[
\therefore I = e^{2x} \sin(x) - 2 \int e^{2x} \sin(x) \, dx
\]

We were rather hoping for an answer, but what we seem to have is another ‘integration by parts’ problem embedded into our development. Don’t worry, that’s quite normal for this type of problem. All we do is attack that last part with ‘integration by parts’ once more, using square brackets. Our desired result will then appear...

\[
I = e^{2x} \sin(x) - 2 \left[ e^{2x} (-\cos(x)) - \int -\cos(x) 2e^{2x} \, dx \right]
\]

Expanding these terms gives...

\[
I = e^{2x} \sin(x) + 2e^{2x} \cos(x) - 4 \int e^{2x} \cos(x) \, dx
\]

That last integral is the same as \( I \), which is really handy, so we may now write...

\[
I = e^{2x} \sin(x) + 2e^{2x} \cos(x) - 4I
\]

\[
\therefore 5I = e^{2x} \sin(x) + 2e^{2x} \cos(x)
\]

If we divide both sides by 5 then we have our final answer...

\[
I = \frac{e^{2x} \sin(x) + 2e^{2x} \cos(x)}{5}
\]

**Question**

Q3.25 Determine: \( \int e^{4x} \sin(2x) \, dx \)

**ANSWER**

\[
Q3.25 \quad \frac{4e^{4x} \sin(2x) - 2e^{4x} \cos(2x)}{20}
\]
3.4 Solution of Engineering Problems with Calculus

3.4.1 Charging RC Circuit

For the circuit below, use Calculus to find the charge stored in the capacitor 2 seconds after the switch closes. Assume zero charge on the capacitor before this event.

For a capacitor charging via a DC source voltage, through a series resistor, the formula for instantaneous current is...

$$i = \frac{E}{R} \cdot e^{-\frac{t}{RC}} \text{ Amps}$$

Since current is the defined as the rate of change of charge ($q$, carried by passing electrons) we may say...

$$i = \frac{dq}{dt}$$

If we integrate both sides with respect to $t$ we will have...

$$\int i \, dt = \int \frac{dq}{dt} \, dt = \int dq = q$$

So we may say that...

$$q = \int i \, dt$$

If we wish to find the amount of charge stored within a certain time period (i.e. 2 seconds in this example) we write...

$$q(0\rightarrow t) = \int_0^t i \, dt = \int_0^2 \frac{E}{R} \cdot e^{-\frac{t}{RC}} \, dt$$

$$= \frac{E}{R} \left[ e^{\left(\frac{-1}{RC}\right)t} \right]_0^2 = \frac{-ERC}{R} \left[ e^{\left(-\frac{t}{RC}\right)} \right]_0^2 = -EC \left[ e^{\left(-\frac{t}{RC}\right)} \right]_0^2$$

Since we know that $R = 1M\Omega$ and $C = 1\mu F$ then $RC = 1 \times 10^6 \times 1 \times 10^{-6} = 10^{6-6} = 10^0 = 1$ we may write...
\[ q_{0-2} = -EC \left[ e^{\left(-\frac{t}{1}\right)} \right]^2_0 = -EC[e^{-t}]^2_0 = -EC[e^{-2} - e^0] = -EC[0.135 - 1] \]

We also know that the DC supply voltage \((E)\) is 10V...

\[ ∴ q_{0-2} = -10 \times 1 \times 10^{-6}[-0.865] = 8.65 \times 10^{-6} \text{ Coulombs} = 8.65 \mu C \]

The plot below illustrates the integration we have just performed...

Current is on the vertical axis and time on the horizontal. When we multiply current by time we get charge, which is shown as the shaded area between 0 and 2 seconds. You may like to experiment with the Graph simulator to produce similar results.

3.4.2 Energising Inductor
Consider the series RL circuit below...

If \(E = 10 \, V\), \(R = 1MΩ\) and \(L = 200mH\), use Calculus to determine the voltage across the inductor after 0.1μs. Use the following formulae in your solution development...

\[ i = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right) \text{ and } v_L = L \frac{di}{dt} \]
We start with...

\[ v_L = L \frac{di}{dt} = L \frac{d}{dt} \left( \frac{E}{R} \left( 1 - e^{-\frac{Rt}{L}} \right) \right) \]

\[ = \frac{LE}{R} \cdot \frac{d}{dt} \left( 1 - e^{-\frac{Rt}{L}} \right) = \frac{LE}{R} \left( \frac{R}{L} \cdot e^{-\frac{Rt}{L}} \right) = E e^{-\frac{Rt}{L}} \]

Putting in the values for E, R, t and L gives...

\[ v_L = 10e^{\left( \frac{-10^6 \times 0.1 \times 10^{-6}}{0.2} \right)} = 6.065 \text{ volts} \]