

Pearson BTEC Levels 4 and 5 Higher Nationals in Engineering (RQF)

Unit 3: Engineering Science (core)

Unit Workbook 4

in a series of 4 for this unit

Learning Outcome 4

Circuit Theory and Electromagnetic Principles

4.1 DC Electrical Principles

4.1.1 Review of Resistors in Series and Parallel

Series Resistors

When resistors are connected in series then each resistor contributes its own barrier to the flow of electrons in a circuit. This is much akin to placing weights along the length of a garden hosepipe: the more weights there are along the length of the hose, and the heavier they become, will collectively contribute to constricting the overall flow of water in the pipe.

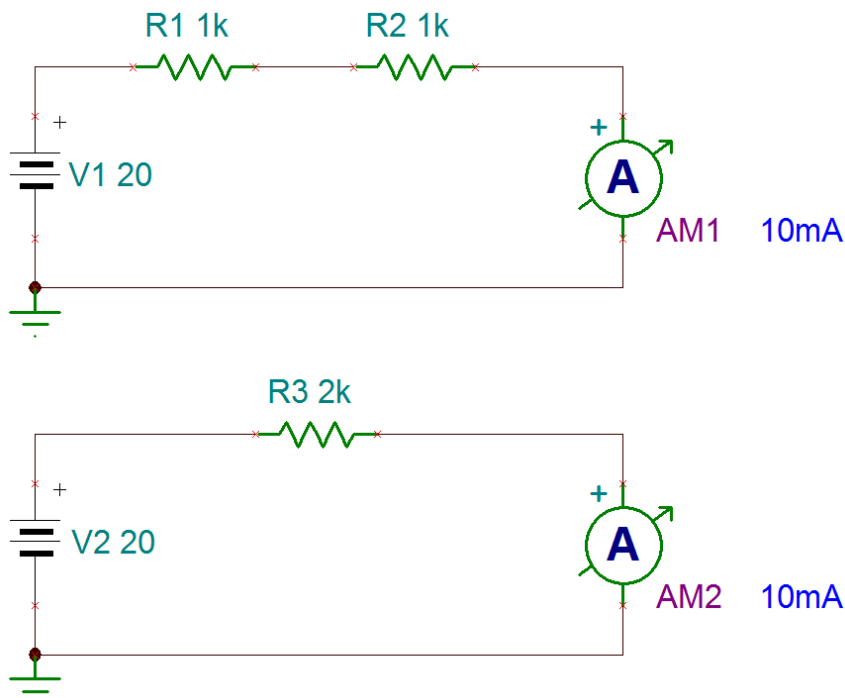
The simple rule for resistors in series is therefore to add their values to find the overall resistance.

$$R_T = R_1 + R_2 + \dots \quad [\Omega]$$

Very straightforward. The first circuit below is drawn in the TINA-TI simulator. It has two 1kΩ resistors connected in series to a 20V DC source. The total resistance is 2kΩ and therefore, from Ohms Law...

$$I = \frac{V}{R_T} = \frac{20}{2000} = 0.01A = 10mA$$

The simulation in the first circuit clearly shows 10mA on the Ammeter.



The second circuit lumps together the two 1kΩ resistors, making a 2kΩ resistor, with exactly the same result. Try it yourself (notice that the earth is needed to reduce errors/warnings in the simulator).

Parallel Resistors

The picture is a little more involved when we come to consider resistors in parallel. The parallel arrangement provides 'alternative paths' for the electrons to negotiate. The more alternative paths there are will mean an easier route for the electrons. A good analogy might be to think of multiple hosepipes connected to a single tap: there will be an increased water flow compared to the single hose arrangement.

To evaluate the overall resistance of a parallel arrangement of resistors we need to consider these multiple paths and use the equation...

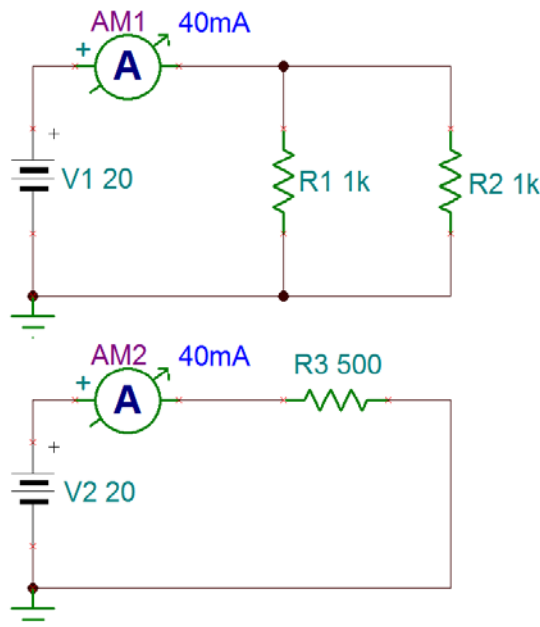
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots \quad [S]$$

Notice the units used in the equation. They are Siemens (reciprocal of resistance) which is the unit used for Conductance. This equation must be used for all cases where there are three or more resistors in parallel. When there are only two resistors in parallel then we may take a common denominator and end up with the product over sum for the overall resistance...

$$R_T = \frac{R_1 R_2}{R_1 + R_2} \quad [\Omega]$$

The first circuit below shows two 1kΩ resistors in parallel, connected to a 20V DC source. Using the product over sum formula for the overall resistance yields...

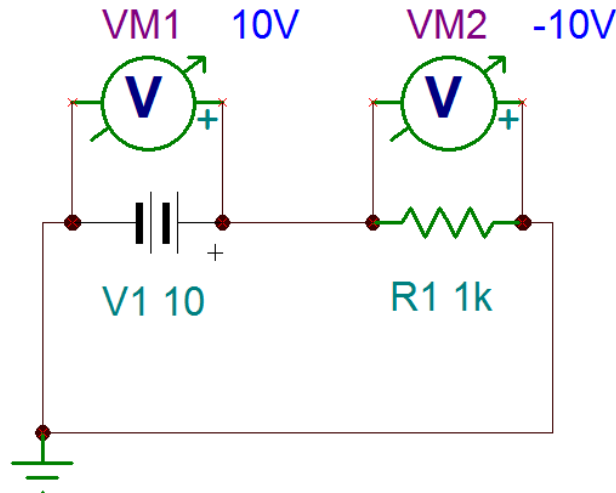
$$R_T = \frac{1000 \times 1000}{1000 + 1000} = \frac{1000000}{2000} = 500\Omega$$



The lower circuit shows that the same current flows using the equivalent resistance of 500Ω.

4.1.2 Kirchhoff's Voltage Law (KVL)

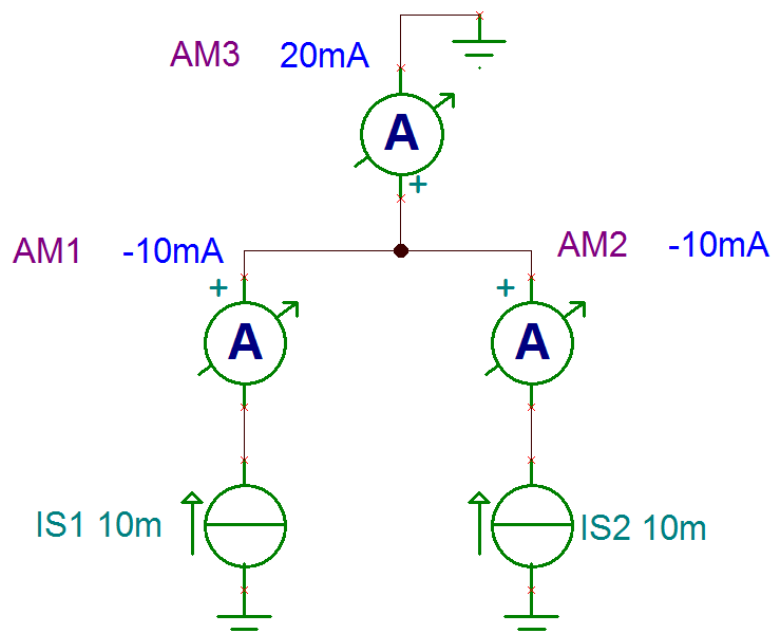
This states that the **algebraic** sum of voltages in any closed loop is zero. For a very simple circuit that means that if we place a resistor across a battery then the algebraic sum of voltages in the closed loop will be zero. Let's have a look at this arrangement...



The crucial part in this circuit is that we have connected the voltmeters in such a way that their '+' terminals both face the same way. Adding +10V and -10V gives us zero volts. This proves KVL in this simple case. Perhaps you would like to construct an arrangement with more resistors and prove that KVL still applies?

4.1.3 Kirchhoff's Current Law (KCL)

This states that the **algebraic** sum of currents at a junction is zero. Let us again use the TINA simulator to prove this...



Here we have ensured that the Ammeters measure current leaving the junction in each case (notice all the Ammeter '+' terminals are connected to the junction). Adding those three currents gives zero. This proves KCL in a simplistic case. Perhaps you would like to construct a more involved arrangement and prove that KCL still applies?

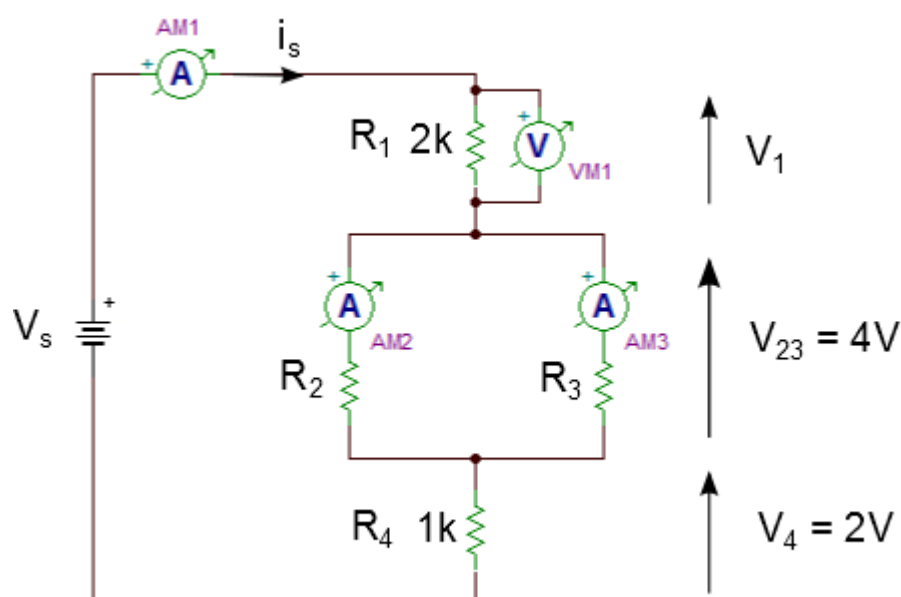
NOTE: The two symbols at the bottom of the above diagram (marked IS1 and IS2) are constant current generators. They do what their title suggests: provide a constant current.

Now that KVL and KCL have been reviewed we are in a position to look through a couple of worked examples.

Worked Example 1

For the DC network given below:

- Calculate the value of the supply current (i_s).
- Determine V_1 .
- Given that $R_2 = R_3$ calculate the current through each of these two resistors.
- Determine the value of the supply voltage (V_s).

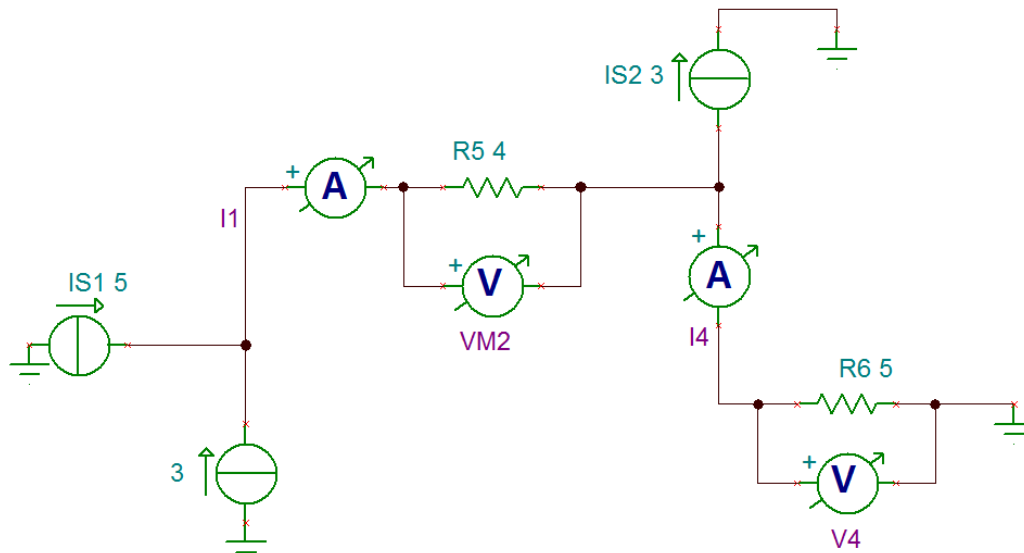


- We can see that the value of R_4 is known, as is the voltage across R_4 . The current i_s must therefore be $2/1000 = 2mA$.
- The supply current also flows through R_1 therefore the voltage V_1 must be $2mA \times 2000 = 4V$.
- Since $R_2 = R_3$ then i_s must split evenly between these two resistors. These currents are therefore $1mA$ each.

- d) Knowledge of KVL tells us that the supply voltage V_s must be equal to the sum of V_1 , V_{23} and V_4 , which is 10V.

Worked Example 2

Consider the circuit below:

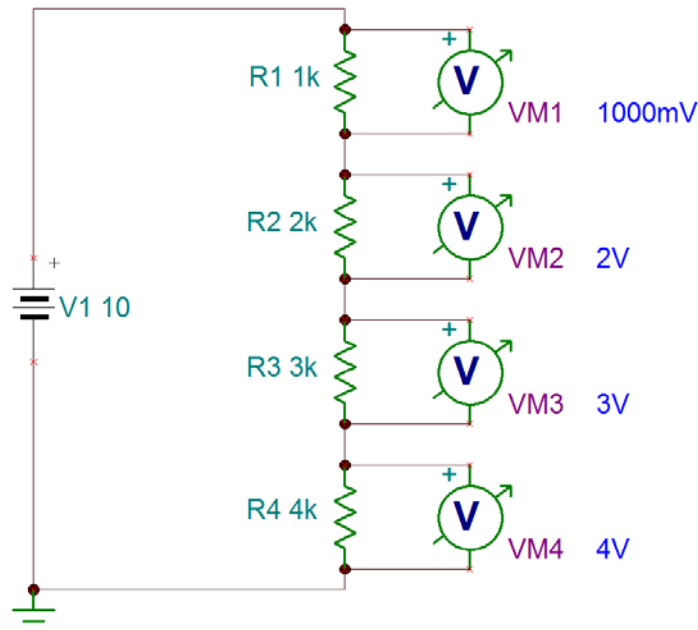


- a) Calculate the value of i_1 and hence V_1
b) Calculate the value of i_4 and hence V_4

- a) Knowledge of KCL tells us that on the lower left of the circuit there are two constant current generators feeding a junction. Since each of these currents must emerge from that junction then the current i_1 must be $(5 + 3) = 8A$.
- b) Looking at the junction to the top right of the circuit we see that 8A (i_1) goes in and 3A comes out at the top. Therefore the current coming out at the bottom of this junction must be $(8 - 3) = 5A$. Since 5A flows through R_6 (which is given as 5Ω) then V_4 must be $(5 \times 5) = 25V$.

4.1.4 Voltage Divider

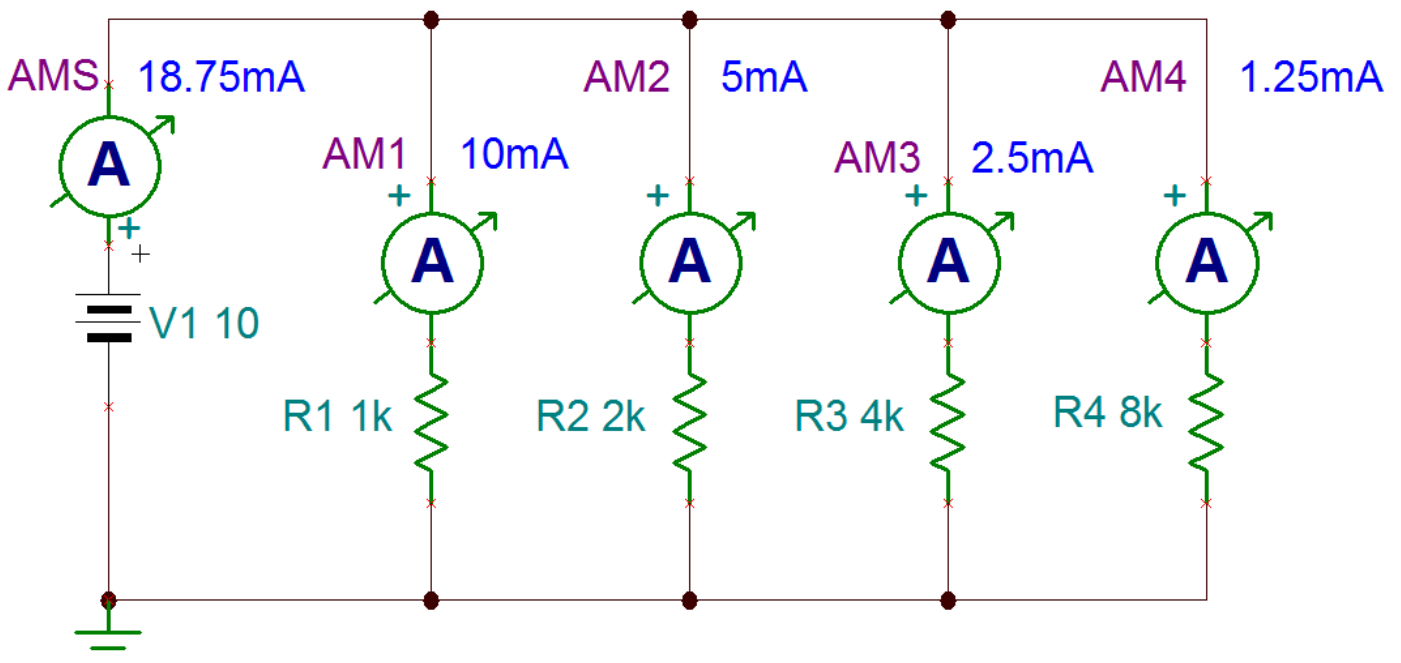
A voltage divider 'divides' a supply voltage across series connected resistors according to the proportion of each resistor to the total resistance seen by the voltage source. Here's the picture...



Clearly the total resistance seen by the 10V battery is 10k Ω . This obviously gives a current of 1mA. Since 1mA flows through each resistor it is a simple matter to apply Ohm's Law to each resistor to calculate the voltage across it. For example, R3 is 3k Ω and 1mA flows through it, so it develops a voltage of 1mA x 3000 = 3V, as seen on the simulator. Try applying KVL to this circuit with your simulator.

4.1.5 Current Divider

A current divider will 'divide' the supply current to each resistor in inverse proportion to the resistance of each branch in comparison to the total resistance seen by the source. Let's take a look at such a circuit...



The total resistance seen by the battery may be calculated as...

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$
$$\therefore \frac{1}{R_T} = \frac{1}{1000} + \frac{1}{2000} + \frac{1}{4000} + \frac{1}{8000} = \frac{8 + 4 + 2 + 1}{8000} = \frac{15}{8000}$$
$$\therefore R_T = \frac{8000}{15} = 533.333\Omega$$

The battery current is then calculated as $10/533.333 = 18.75\text{mA}$. This figure is confirmed by the source Ammeter (AMS). As you can see, the sum of the resistor currents equals 18.75mA also, as expected. The key to working out the currents in this circuit is to notice that each resistor is presented with 10V by the battery. The calculations are then easy.

4.1.6 DC Motor and Generator Principles

Video

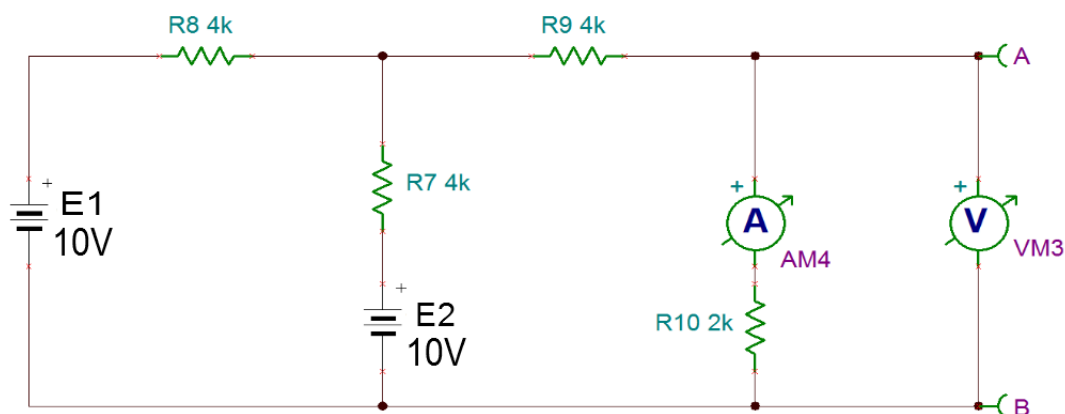
This classic YouTube video discusses DC motors and generators much better than any workbook can. Do take the time to review it.

4.1.7 Superposition Theorem

This states that in a circuit with more than one voltage source the overall circuit voltages and currents may be calculated by adding the contributions from each individual voltage source. To find the contributions from one individual voltage source we must short circuit all of the other voltage sources. This all best explained with an example...

Worked Example 3

For the circuit below, employ the Superposition Theorem to calculate the current through R_{10} and the voltage V_{AB} .

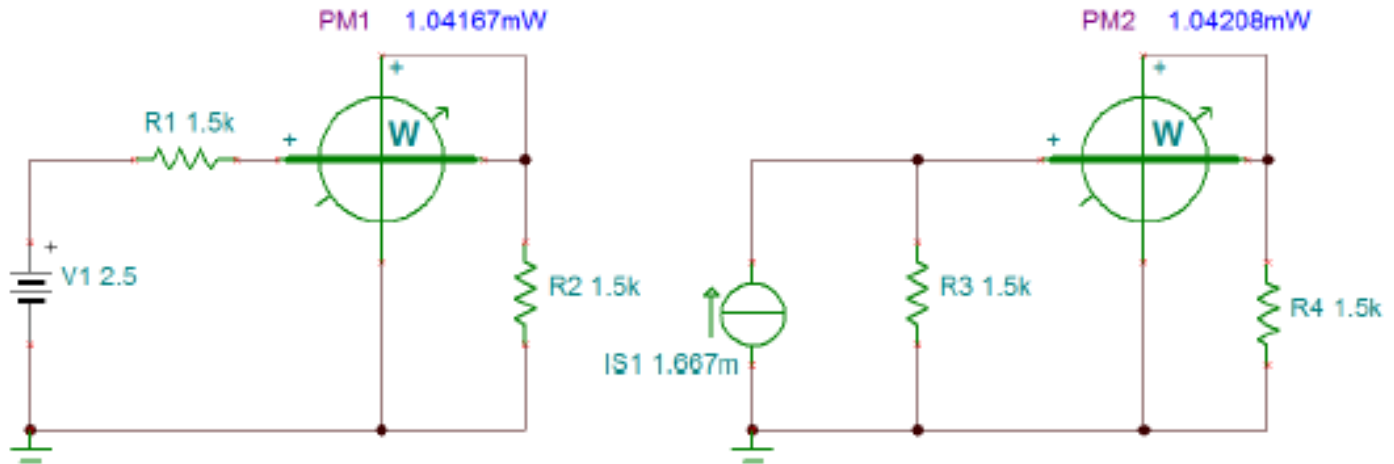


We shall begin by considering the contribution from e.m.f (battery) E_1 alone. To do this we must replace battery E_2 with a short circuit. The resulting circuit is drawn below...

What we are expecting for this load resistor power will be the current squared times R_L , right? Since the total resistance seen by the battery will be $3k\Omega$ then the current will be $2.5/3000 = 0.833333mA$. The load power is then given by...

$$P_L = I^2 R_L = 0.000833333^2 \times 1500 = 1.041mW$$

Let's use the TINA-TI simulator to see if this power calculation is correct...



On the left is our Thévenin circuit, where the power meter measures 1.04167mW. On the right is our Norton circuit, where the power meter measures 1.04208mW. This is only a tiny discrepancy and is only due to rounding in our previous calculations.

It will be very useful for you to reproduce these results in the TINA-TI simulator yourself.

4.1.11 Fundamental Passive Components

Resistors, capacitors and inductors are our three fundamental passive components in electrical engineering. Resistors are straightforward, in that the voltage across a resistor is always in phase with the current flowing through it. This is not the case for capacitors or inductors.

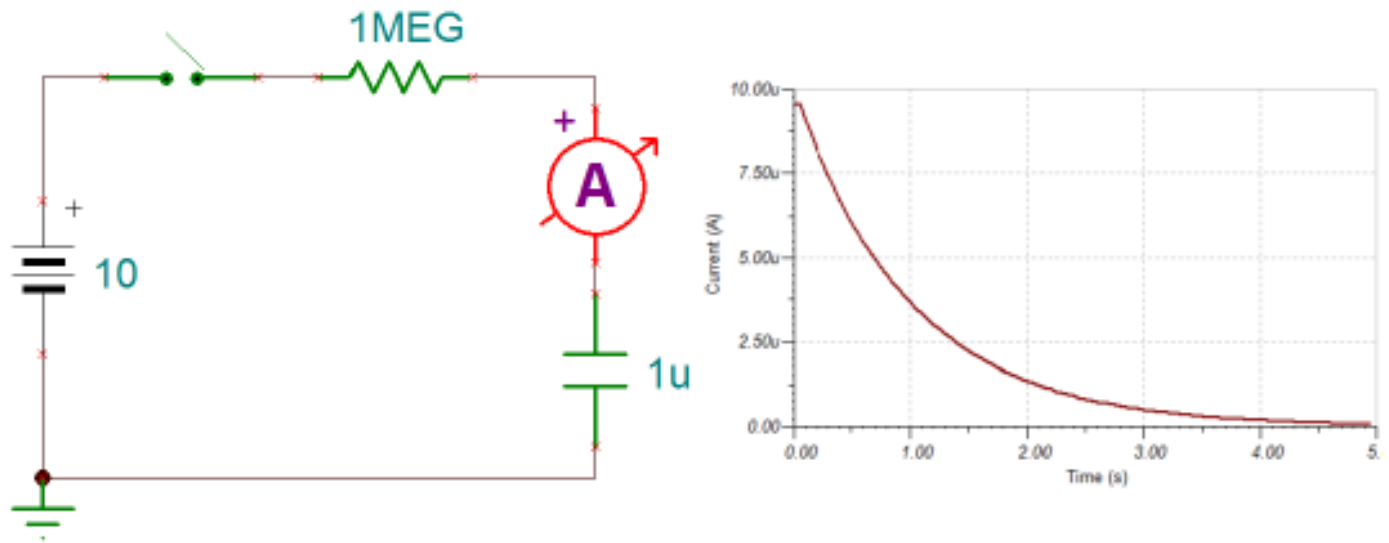
Capacitors take time to charge (and discharge), just like a bathtub takes time to fill (and empty). The energy stored in a capacitor is in the form of an electrical field formed between the plates.

Inductors do not store electrical charge. What they do store, however, is energy in the form of an electromagnetic field around the coil. This field is not built instantly – it takes time, just like trying to collapse the field takes time.

Capacitor-Resistor (CR) and Inductor-Resistor (LR) circuits are series circuits which are useful for timing and filtering. Our next section examines these circuits in some useful detail.

4.1.12 CR Circuits

To charge a capacitor via a series resistor takes a finite amount of time. Let's look at a TINA simulation for such a situation...



Once the switch is closed the 'transient analysis' shown on the curve is produced. This curve represents current on the vertical axis and time on the horizontal axis. The maximum current shown is 10µA. This current happens at the instant the switch is closed. As time goes on the current reduces towards zero, as shown. There is a formula which we use for calculating this current at various points in time...

$$i = \frac{E}{R} e^{-t/RC} \text{ [A]}$$

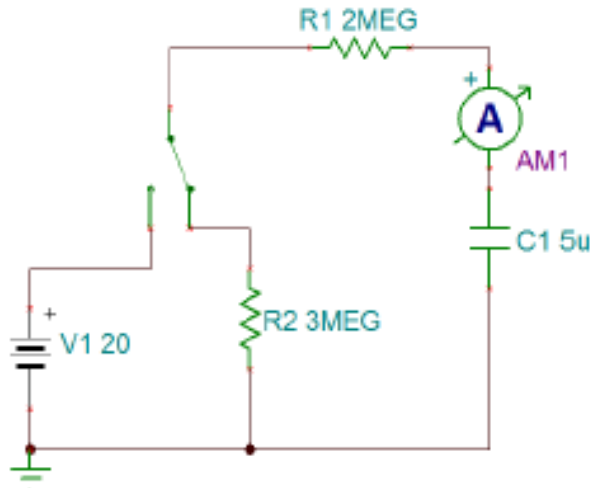
Here, E represents the battery voltage, R is the resistor, e is the natural constant, t is time and C is the value of the capacitor. The current curve exhibits what is known as 'exponential decay' because of the influence of e. Let's do a quick check with this equation - the curve looks like it shows around 3.7µA when time is 1 second. Plugging the component values and t = 1 into the formula yields...

$$i = \frac{E}{R} e^{-t/RC} = \frac{10}{1,000,000} e^{-1/1,000,000 \times 0.000001} = 10^{-5} e^{-1} = 3.68\mu\text{A}$$

This is in very good agreement with what we see on the curve. Let's look at a worked example involving the charge and discharge of a capacitor.

Worked Example 6

Consider the circuit below. We see that if the switch is moved to the left then we place a 2MΩ resistor in series with a 5µF capacitor. If the switch is moved back to the right then our charged capacitor is discharged via a series resistance of 5MΩ (R1 + R2).



Assuming that there is no initial charge stored in the capacitor and the switch is suddenly moved to the left, calculate:

- The capacitor current after 10s.
- The voltage across R_1 after 40s.

Now assume that the capacitor has reached full charge. The switch is then moved to the right. Calculate:

- The capacitor current 25s after switching to the right.
- The voltage across R_2 100s after switching to the right.

- The capacitor is charged via R_1 . The capacitor current is also the current through R_1 , since we have a series circuit. We can use our formula for current, as before...

$$i = \frac{E}{R} e^{-t/RC} = \frac{20}{2,000,000} e^{-10/2,000,000 \times 0.000005} = 10^{-5} e^{-1} = 3.68 \mu A$$

- To know the voltage across R_1 after 40 seconds we need to know the current after 40 seconds...

$$i = \frac{E}{R} e^{-t/RC} = \frac{20}{2,000,000} e^{-40/2,000,000 \times 0.000005} = 10^{-5} e^{-4} = 0.183 \mu A$$

The voltage across R_1 after 40 seconds is therefore $0.183 \mu A \times 2M\Omega = 0.366V$

- The capacitor is fully charged to 20V and acts as the source of voltage (i.e. like battery E). The capacitor is in series with $5M\Omega$ (i.e. $R_1 + R_2$). We may again use our formula for current...

$$i = \frac{E}{R} e^{-t/RC} = \frac{20}{5,000,000} e^{-25/5,000,000 \times 0.000005} = 4 \times 10^{-6} \times e^{-1} = 1.47 \mu A$$

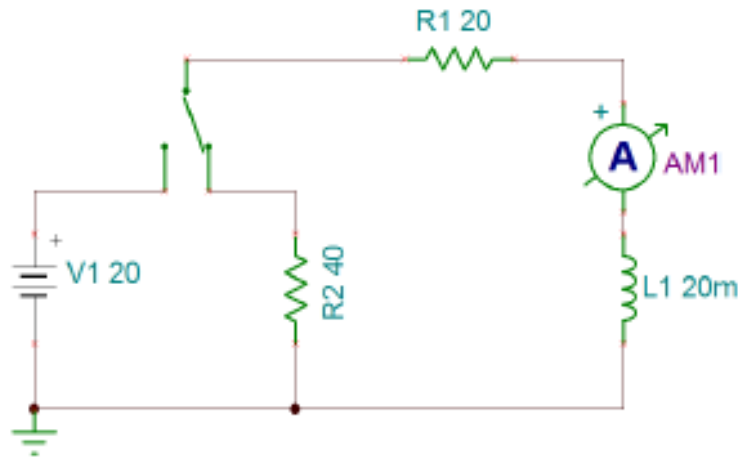
- We need the current after 100 seconds...

$$i = \frac{E}{R} e^{-t/RC} = \frac{20}{5,000,000} e^{-100/5,000,000 \times 0.000005} = 4 \times 10^{-6} \times e^{-4} = 73.3nA$$

The voltage across R_2 after 100 seconds is therefore $73.3nA \times 3M\Omega = 0.22V$

Worked Example 7

Consider the LR circuit below:



Assuming that there is no initial energy stored in the inductor and the switch is suddenly moved to the left, calculate:

- The inductor current after 1ms.
- The voltage across R_1 after 4ms.

Now assume that the voltage across R_1 has reached its maximum value of 20V. The switch is then moved to the right. Calculate:

- The inductor current 333 μ s after switching to the right.
 - The voltage across R_2 1.333ms after switching to the right.
- a) The coil is energised via R_1 . The coil current is also the current through R_1 , since we have a series circuit. We can use our formula for current, as before...

$$i = \frac{E}{R}(1 - e^{-Rt/L}) = \frac{20}{20}(1 - e^{-20 \times 0.001/0.02}) = 1 \times (1 - e^{-1}) = 632mA$$

- b) To know the voltage across R_1 after 4ms we need to know the current after 4ms...

$$i = \frac{E}{R}(1 - e^{-Rt/L}) = \frac{20}{20}(1 - e^{-20 \times 0.004/0.02}) = 1 \times (1 - e^{-4}) = 0.98A$$

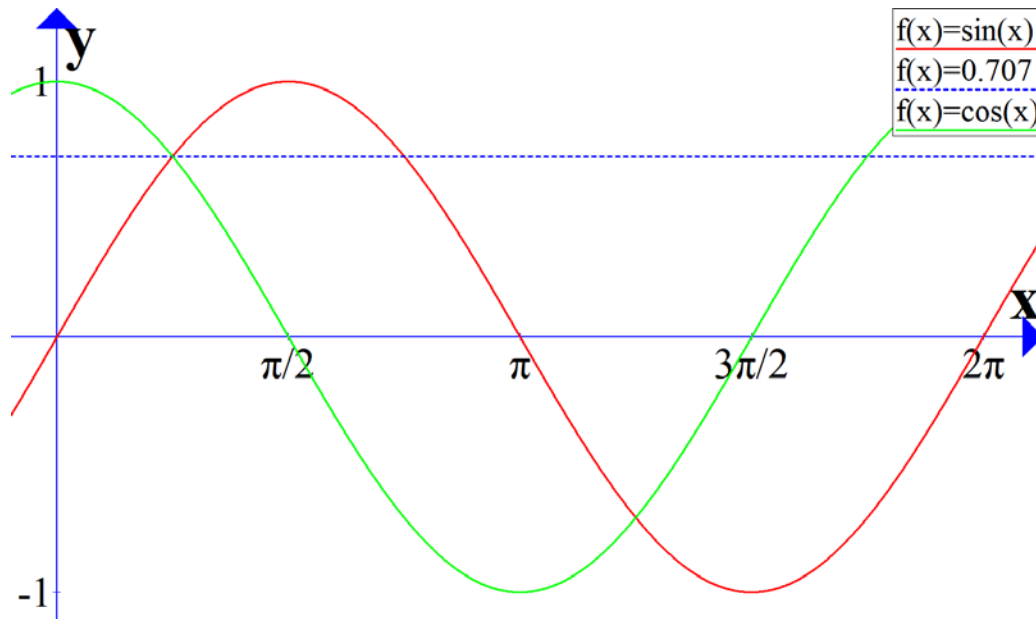
The voltage across R_1 after 4ms is therefore $0.98A \times 20\Omega = 19.6V$

- c) The coil is now fully energised and acts as a source of power (i.e. like battery E). The capacitor is in series with 60Ω (i.e. $R_1 + R_2$). Clearly, the energy stored in the electromagnetic field of the coil will be expended by providing current to R_1 and R_2 , so once it's gone it's gone. The situation we have is one of exponential decay in the current. We may use our decay formula for current in a series RL circuit...

4.2 Complex Waveforms

4.2.1 Features of Sinusoidal Waveforms

Sinusoidal voltages and currents complete a full period (cycle) in 2π radians (360 degrees). A basic sine wave is shown in red below. Also shown (green) is a cosine wave which leads the sine wave by $\pi/2$ radians (90 degrees). Both are sinusoidal in nature/shape, the difference is in the phase (i.e. cosine leads).



We can scale the basic amplitude (strength) of these waves from 1 to any value we wish. For example, the UK mains has a peak value (from 0 all the way to the top of the crest) of 325.2V. This is not a value we normally associate with the UK mains, we are used to seeing 230V. The 230V figure is derived from the root-mean-square (RMS) value of the sinusoid and is the equivalent voltage that a DC source might have to supply to deliver the same power. This RMS value comes out to be 0.707 times the peak value, so 0.707×325.2 gives around 230. This 0.707 level of the 1V basic sinusoid is shown as the blue dotted line on the figure.

We may also scale the frequency of the wave (i.e. how many cycles should there be in one second). The UK mains has a frequency of 50Hz, of course, so that means there are $1/50 = 20\text{ms}$ in a full cycle. If we want a 1MHz sinewave then it will have a period of $1\mu\text{s}$ (i.e. $1/1,000,000$).

We may also change the phase to any value required. Each of these parameters is very useful to know when we come to realise just how useful sine waves and cosine waves are as basic building blocks for other types of signal.

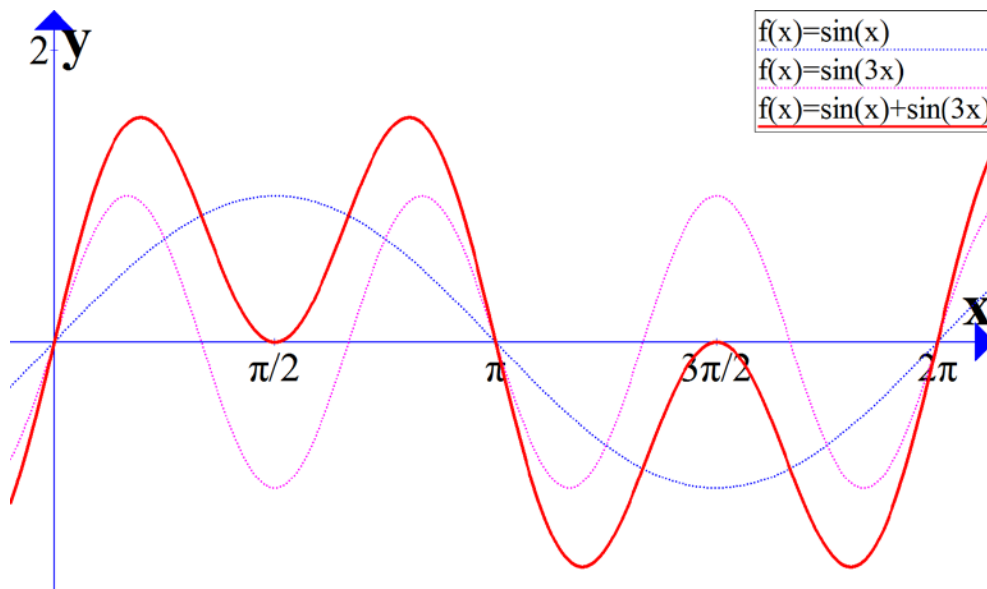
4.2.2 Synthesis of Complex Waveforms

Any periodic waveform, including those with flat portions, may be produced just with combinations of sine waves. The trick is to adjust the magnitude, frequency and phase of these sinusoids to just the right amounts. *This area of study is known as 'Fourier Analysis' and is investigated further in several level 5 units on the higher national specification.*

We start *our* analysis with a sinusoid of a **fundamental** frequency. Integer multiples of this fundamental are known as **harmonics**. For example...

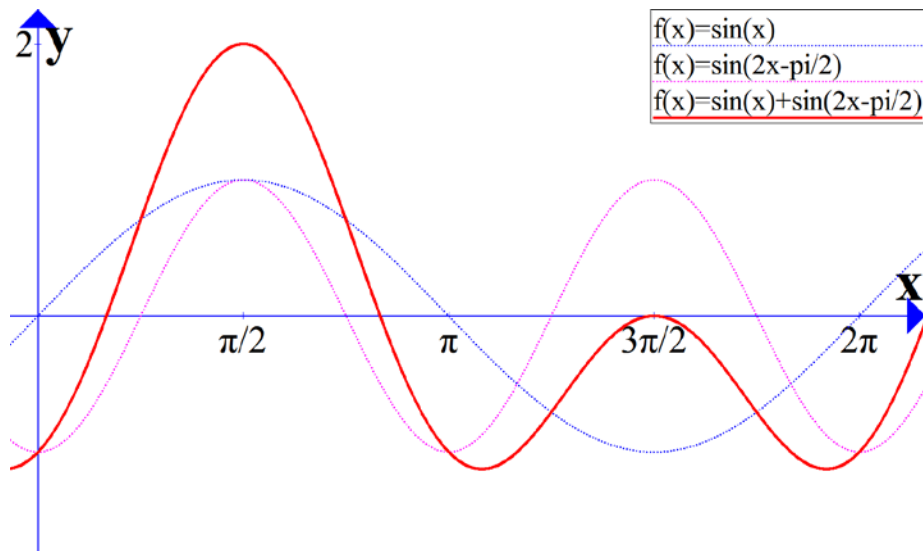
- 1 MHz fundamental
- 2 MHz 2nd harmonic (even)
- 3 MHz 3rd harmonic (odd)
- 4 MHz 4th harmonic (even)
- 5 MHz 5th harmonic (odd)
- 6 MHz 6th harmonic (even)
- 7 MHz 7th harmonic (odd)
- etc. etc.

We therefore have **even harmonics** and **odd harmonics**. Let's take a look at some of the complex waveforms we can build, just with sinusoids...



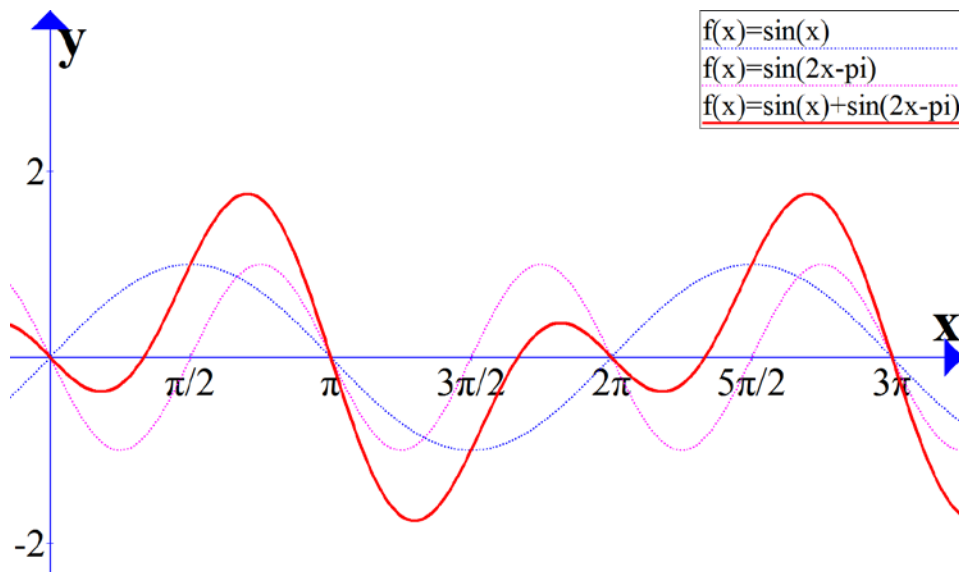
The graphic above shows a fundamental (blue) and its 3rd harmonic (pink). When we add these two waveforms together we produce (synthesise) the complex waveform shown in red. Notice that **this waveform is symmetrical about the horizontal axis**.

What about starting with the fundamental and adding the 2nd harmonic lagging by $\pi/2$ radians (90 degrees)? Here's the plot for that...



Notice that the **positive half-cycles are dissimilar in shape to the negative half-cycles**.

We now look at the new situation where a fundamental is added to its 2nd harmonic which is π radians (180 degrees) out of phase with the fundamental? Here's the plot...



Notice here that **the positive half-cycles are the reverse mirror image of the negative half-cycles**.

You can play about with fundamentals and their harmonics by using the free [Graph software](#), as used above.

4.2.4 RL Circuits

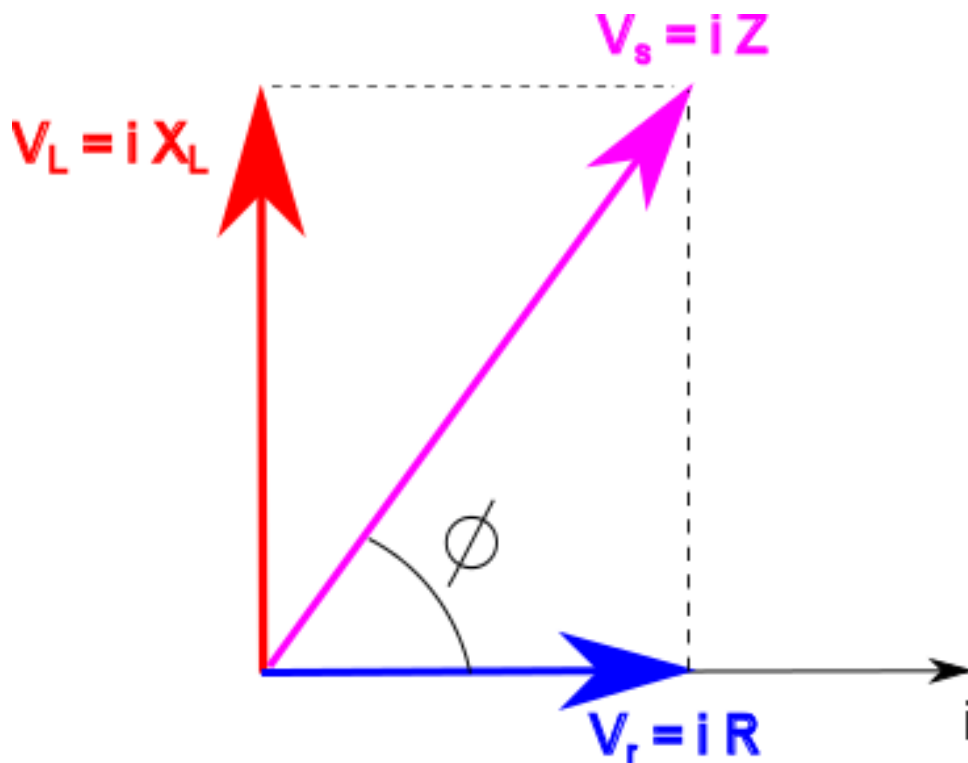
For series RL circuits there are some important changes to the formulae used in the previous section. The reactance of the coil is different to the reactance of a capacitor and uses a different formula. The impedance formula uses X_L rather than X_C and the corner frequency formula is also different. These differences are highlighted below...

$$X_L = 2\pi fL \quad [\Omega]$$

$$Z = \sqrt{R^2 + X_L^2} \quad [\Omega]$$

$$f_c = \frac{R}{2\pi L} \quad [Hz]$$

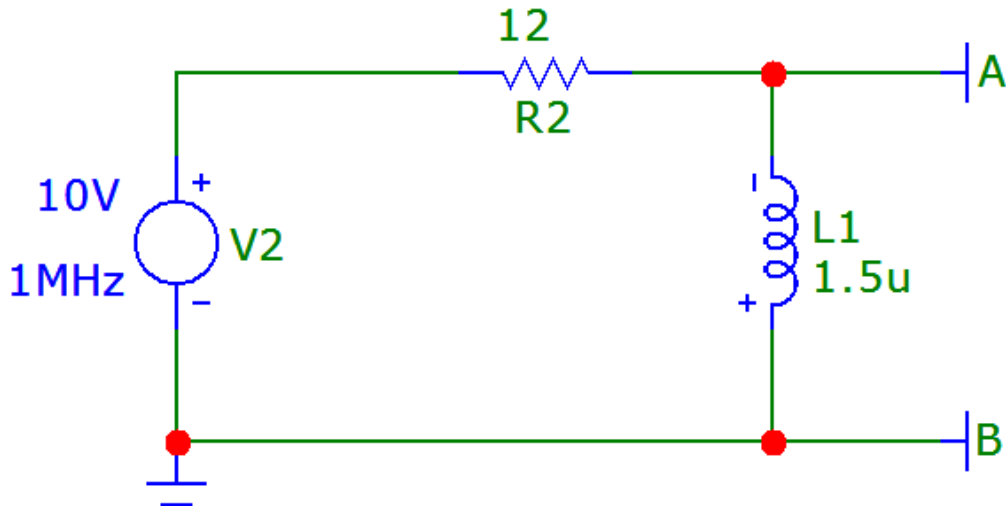
Another difference is in the phasor diagram. Here the voltage across the coil **leads** the current by 90 degrees...



Let's look at a worked example involving an RL circuit.

Worked Example 9

Consider the RL circuit below.



Calculate the:

- Reactance of the coil
- Impedance of the circuit
- Admittance of the circuit
- Power Factor
- Supply Current
- True Power
- Apparent Power
- Half-power Frequency

a) $X_L = 2\pi fL = 2\pi \times 1 \times 10^6 \times 1.5 \times 10^{-6} = 9.42 \Omega$

b) $Z = \sqrt{R^2 + X_L^2} = \sqrt{12^2 + 9.42^2} = \sqrt{144 + 88.7} = \sqrt{232.7} = 15.3 \Omega$

c) $Y = \frac{1}{Z} = \frac{1}{15.3} = 65 \text{ mS}$

d) $PF = \frac{R}{Z} = \frac{12}{15.3} = 0.78$

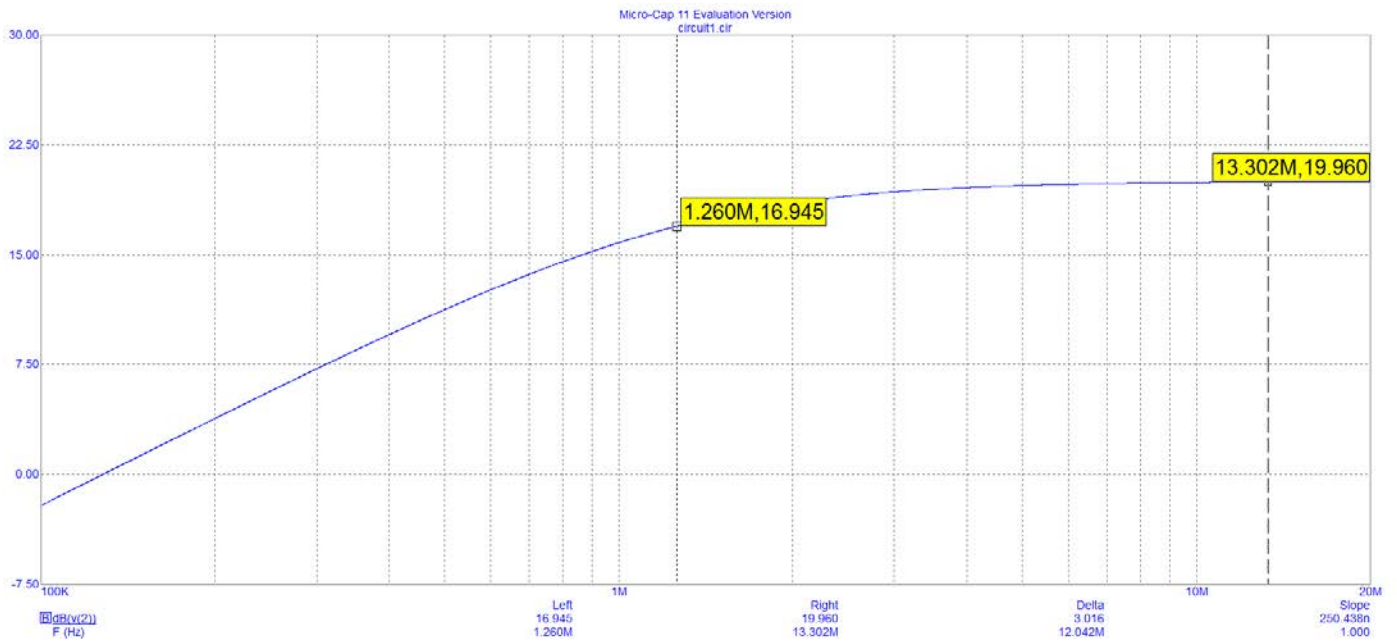
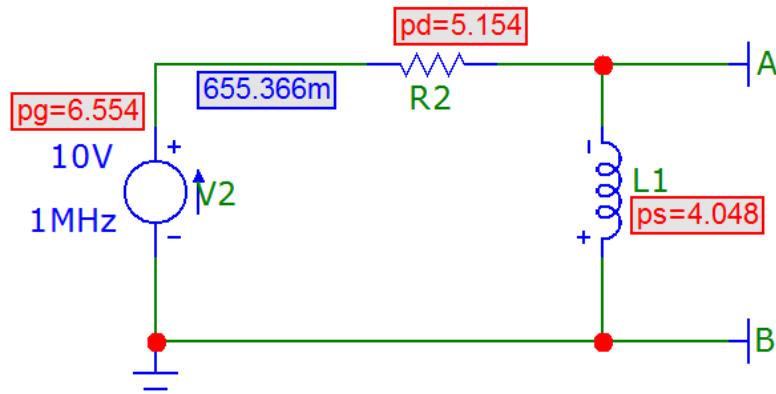
e) $i = \frac{V_s}{Z} = \frac{10}{15.3} = 654 \text{ mA}$

f) $P_{true} = i^2 R = 0.654^2 \times 12 = 5.1 \text{ W}$

g) $P_{apparent} = i^2 Z = 0.654^2 \times 15.3 = 6.5 \text{ VA}$

h) $f_c = \frac{R}{2\pi L} = \frac{12}{2\pi \times 1.5 \times 10^{-6}} = 1.28 \text{ MHz}$

These answers are fairly well confirmed by the MicroCap simulation results below.



It is clear from this plot in the frequency domain that this RL circuit is a High-Pass Filter (HPF).

4.2.5 RLC Circuits

Our analysis here will involve a resistor, capacitor and inductor connected in series. We shall be particularly interested in features of these circuits known as ‘resonant frequency’ and ‘Q-Factor’.

To consider the **resonant frequency** of a series-connected RLC circuit we must account for the reactances of the capacitor and coil. These act in an opposite sense in the circuit (as in the previous phasor diagrams) and when their reactances are equal in magnitude we have resonance in the circuit. When this happens the only barrier to current flow in the circuit is the resistor itself.

Let’s examine the situation where the capacitive reactance equals the inductive reactance...

$$X_C = X_L$$
$$\therefore \frac{1}{2\pi fC} = 2\pi fL$$

We are interested in frequency here, so let’s isolate the f term...

$$1 = 2\pi fL \times 2\pi fC$$
$$\therefore \frac{1}{2\pi L 2\pi C} = f^2$$
$$\therefore f^2 = \frac{1}{(2\pi)^2 LC}$$

Taking the square root of both sides then gives...

$$f = \sqrt{\frac{1}{(2\pi)^2 LC}} = \frac{\sqrt{1}}{\sqrt{(2\pi)^2} \times \sqrt{LC}}$$

We can tidy this expression up a little to yield the formula for resonant frequency...

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Notice that we have changed f into f_0 , which is the standard way to represent the resonant frequency of a circuit.

An important parameter used to measure the selectivity of a tuned (resonant) circuit is the **Q-Factor**. The Q-Factor for a series RLC circuit is given by...

$$Q - Factor = \frac{1}{R} \sqrt{\frac{L}{C}}$$

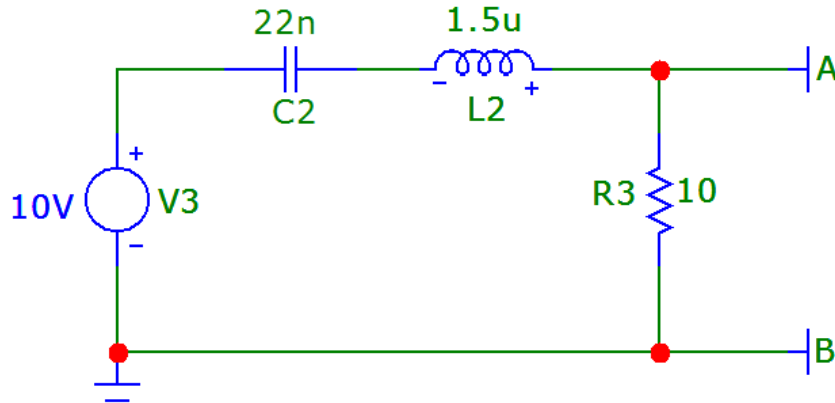
We may relate the resonant frequency, Q-Factor and bandwidth (BW) of a resonant circuit by...

$$f_0 = Q \times BW$$

Let’s look at a worked example involving a series RLC circuit.

Worked Example 10

Consider the series RLC circuit below.



Calculate the:

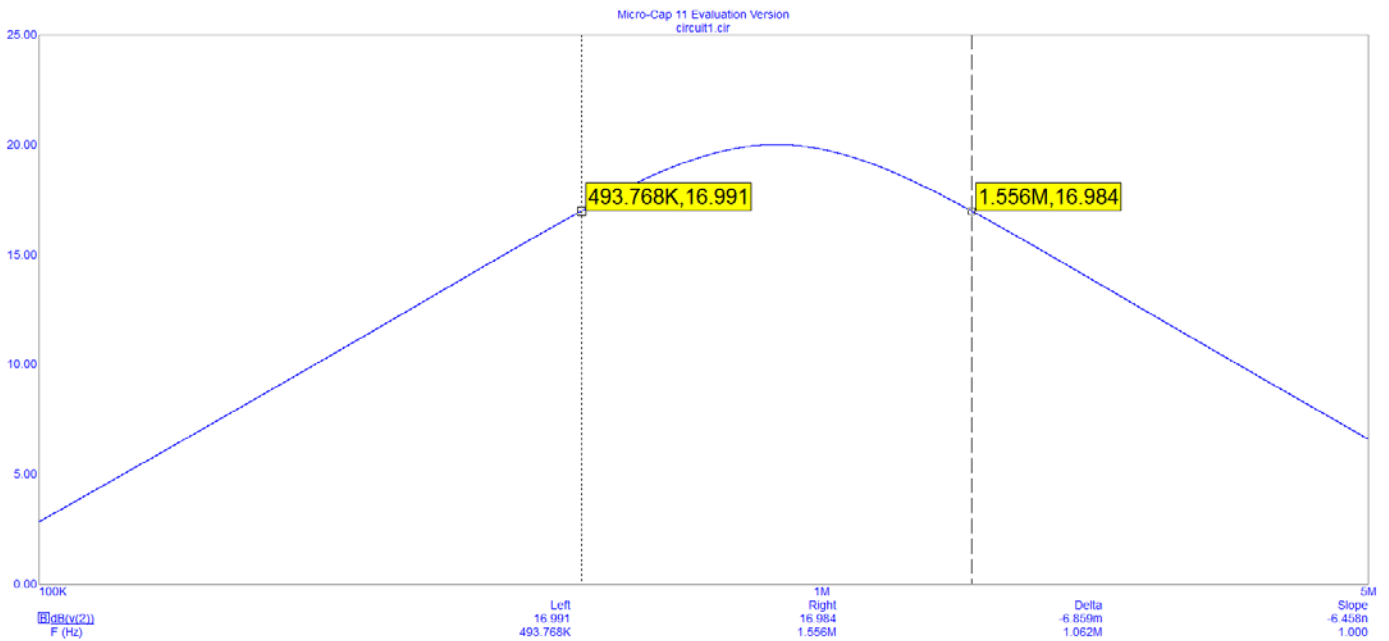
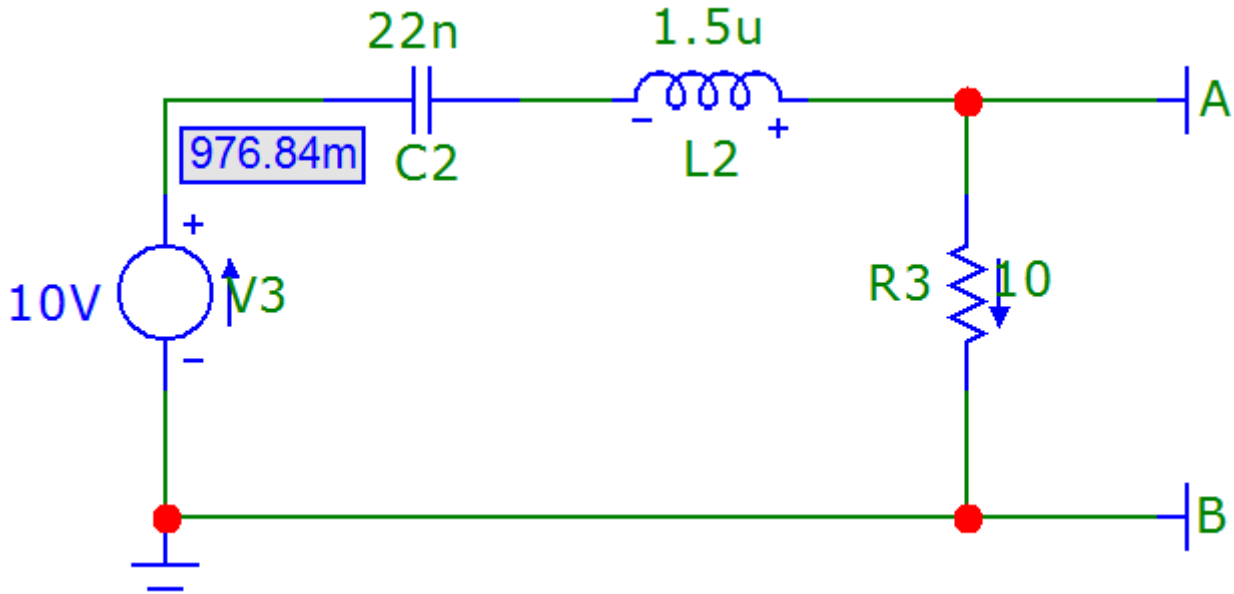
- Resonant Frequency**
- Circuit current if the resonant frequency is used**
- Q-Factor**

$$\begin{aligned} \text{a) } f_0 &= \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{1.5 \times 10^{-6} \times 22 \times 10^{-9}}} = 876 \text{ kHz} \\ \text{b) } i_{res} &= \frac{V_s}{R} = \frac{10}{10} = 1 \text{ A} \\ \text{c) } Q - \text{Factor} &= \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{1}{10} \sqrt{\frac{1.5 \times 10^{-6}}{22 \times 10^{-9}}} = 0.83 \end{aligned}$$

Note that the bandwidth is determined by dropping vertical lines at the -3dB (half power) points on the AC analysis plot. Since we now know the Q-Factor and resonant frequency, we could determine the bandwidth quite easily...

$$f_0 = Q \times BW \quad \therefore \quad BW = \frac{f_0}{Q} = \frac{876,000}{0.83} = 1.05 \text{ MHz}$$

Let's use MicroCap again to confirm the above results...

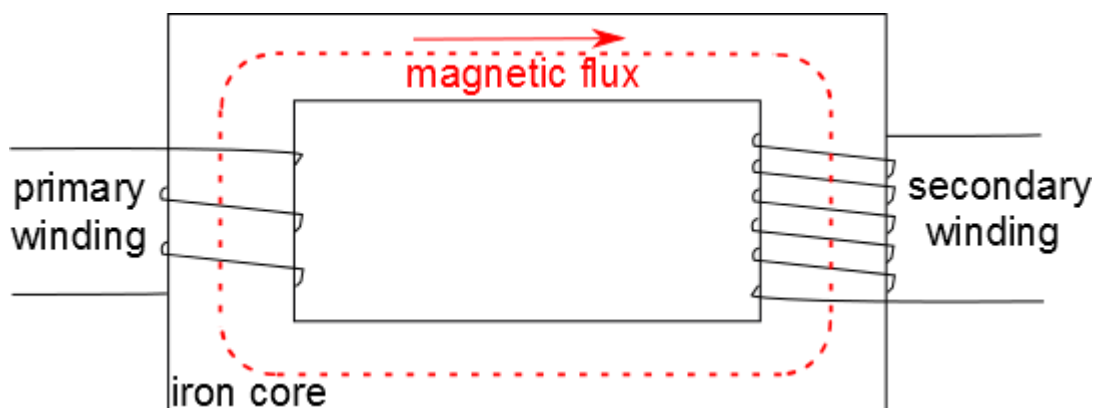


The Delta markers at the bottom of the plot show a bandwidth of 1.06 MHz, which is in good agreement with our calculation.

4.3 Transformers

4.3.1 Transformer Principles

A **transformer** can transfer energy by means of electromagnetic induction. A typical transformer will have a primary winding and a secondary winding, both around a common iron core, as shown below.



When an AC current is applied to the primary winding this generates magnetic flux within the iron core, which links and cuts the windings on the secondary. This concept of magnetic flux linkage produces a voltage on the secondary.

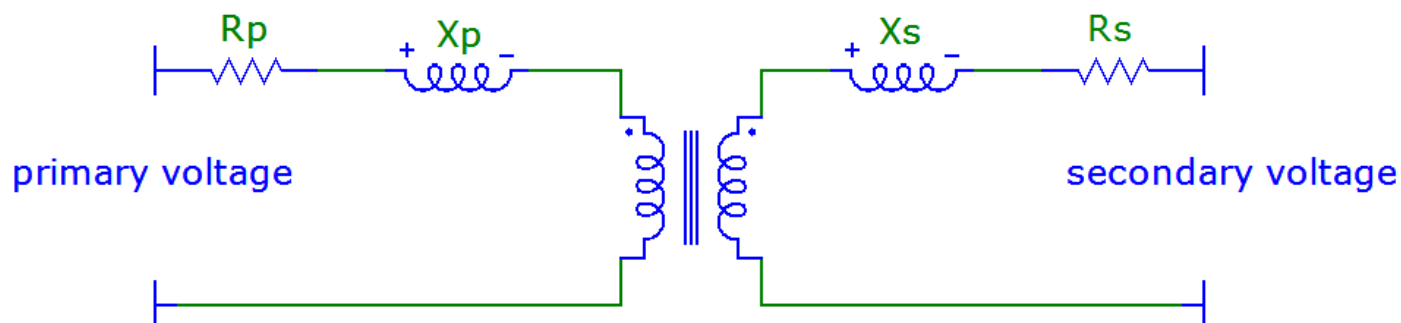
Video 1

This short video introduces you to basic transformer principles

Video 2

Check out types of transformer windings in this short video

A simplified equivalent circuit for a transformer is shown below.



Here the resistance and reactance of the primary and secondary windings are accounted for.