

Pearson BTEC Levels 5 Higher Nationals in Engineering (RQF)

Unit 39: Further Mathematics

Unit Workbook 2

in a series of 4 for this unit

Learning Outcome 2

Matrix Methods

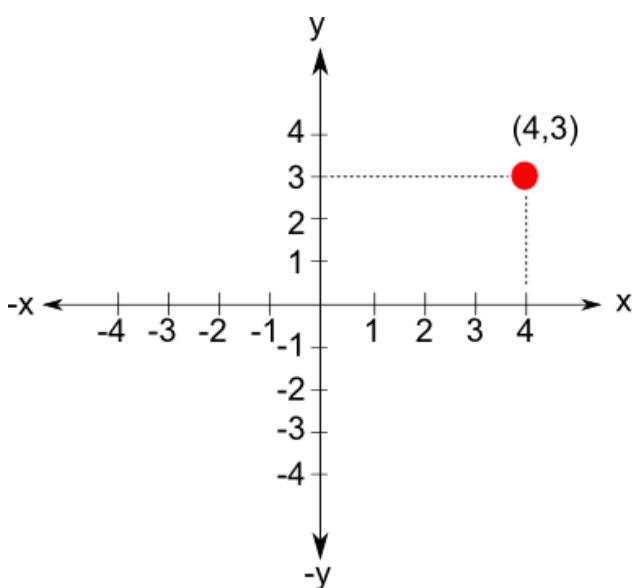
2.1 Vector Notation and Operations

2.1.1 Cartesian Co-Ordinates and Unit Vectors

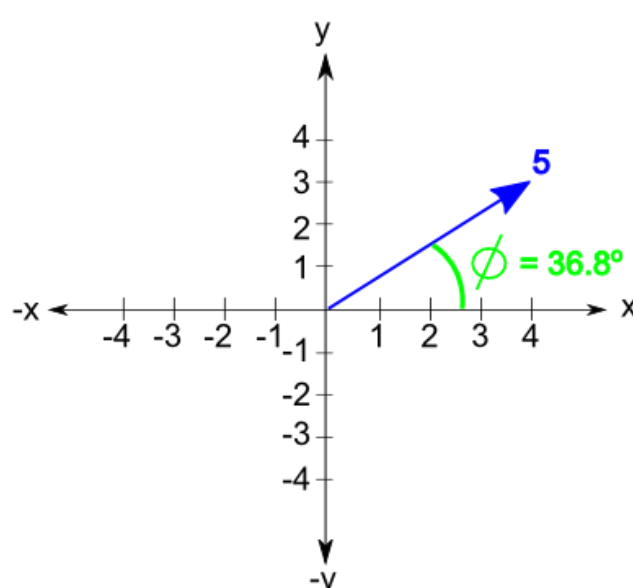
A Scalar is a quantity with magnitude only. Examples of a scalar are an amount of cash or the age of a person. **A Vector has both magnitude and direction.** Examples of a vector are the motion of a car or the magnetic field emanating from a bar magnet.

We shall initially examine vectors in two dimensions (north-south and east-west if you like) and then move on to look at vectors in three dimensions (perhaps inside a cube).

Let's look at the Cartesian co-ordinate system...



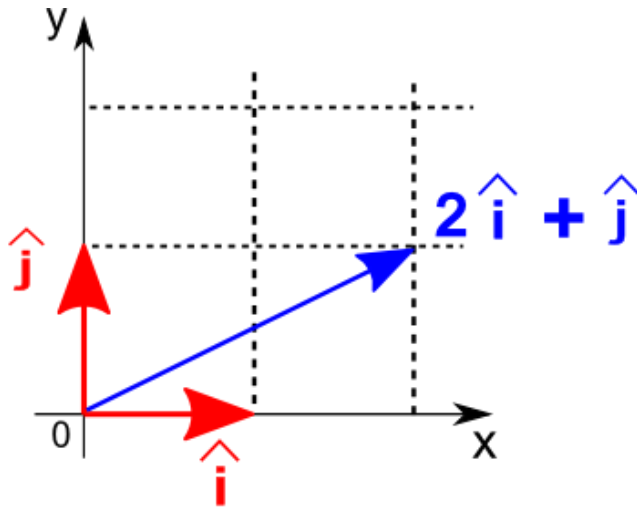
Point on a Cartesian plane



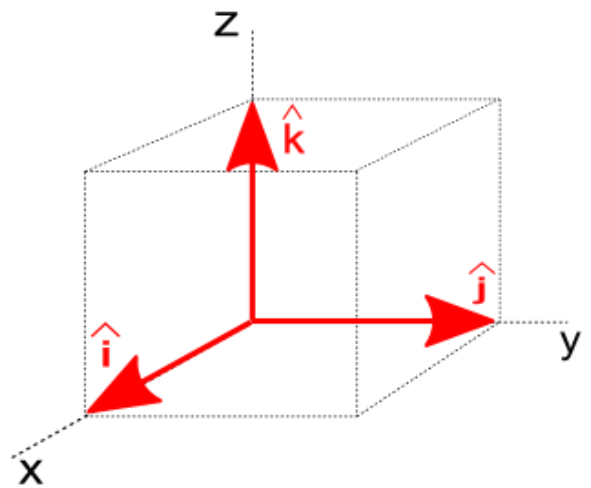
Vector representation of the same point

The point in red sits in a Cartesian co-ordinate system. It has a horizontal component ($x = 4$) and a vertical component ($y = 3$). We may represent the same point as a vector (in blue). A little bit of Pythagoras will yield a length of 5 for this vector and an angle of 36.8° from the horizontal reference plane. Should a car be travelling at 5 mph in a direction 4 miles due east and 3 miles due north then it may be represented as per the figure on the right i.e. a vector quantity.

A Unit Vector is a scaled version of a vector, always having a magnitude (length) of 1. To find the unit vector version of a starting vector we simply divide the vector by its original magnitude. In our example above, the magnitude is 5 so we divide 5 to give a unit vector magnitude of 1. The horizontal component of this unit vector will then be $4/5 = 0.8$ and the vertical component will be $3/5 = 0.6$. A quick check using Pythagoras will give magnitude = $\sqrt{0.8^2 + 0.6^2} = 1$. Unit vectors will be very useful in our calculations, especially when we work in three dimensions. Let's look at a unit vector co-ordinate systems in two and three dimensions...



Unit Vectors along 2-dimensional axes



Unit Vectors along 3-dimensional axes

The 2-D drawing shows our normal x and y axes in black. If we take a vector along the x axis and give it a length of 1 then this will be a unit vector along x which is called \hat{i} . Along the y axis we may produce a unit vector called \hat{j} . The blue vector has a horizontal component of $2\hat{i}$ and a vertical component of $1\hat{j}$ or, just \hat{j} , to save some ink. The 3-D drawing on the right shows these unit vectors again, plus another one (\hat{k}) to represent the third dimension. These unit vectors will be extensively used throughout this workbook and in your third assignment.

2.1.2 Types of Vector and Vector Representation

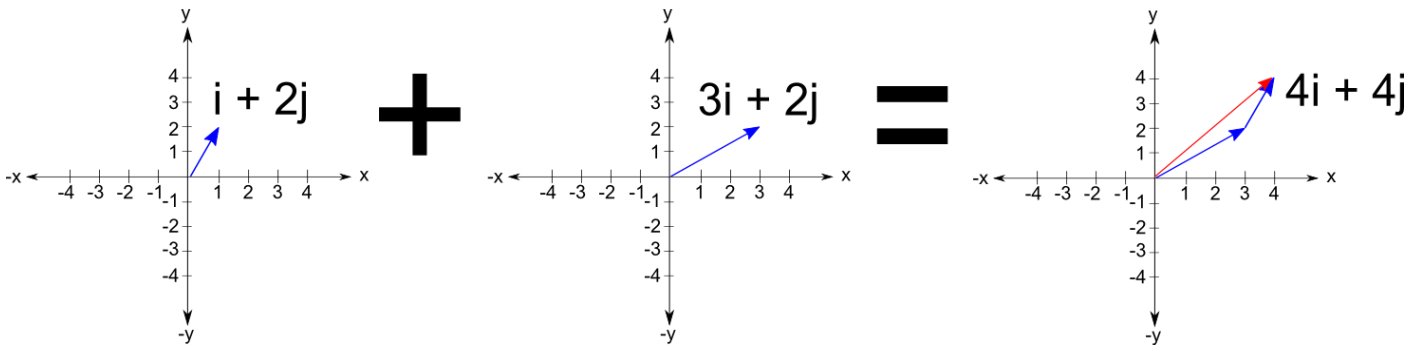
There are many types of vector. Some of the most notable ones are...

- **Co-initial vectors** their starting points are the same
- **Co-terminal vectors** their ending points are all the same
- **Coplanar vectors** they lie in the same plane
- **Null vector** has direction but no magnitude (advanced stuff)
- **Unit vectors** we now know a bit about these
- **Resultant vector** result of adding two vectors together
- **Like vectors** they act in the same direction
- **Unlike vectors** they are parallel but act in opposite directions
- **Axial vectors** act along one of the axes
- **Space vector** exists within 3-D space

Representation of vectors is performed by using multiples of the unit vector along each axis. The previous sketch represented a vector in this way.

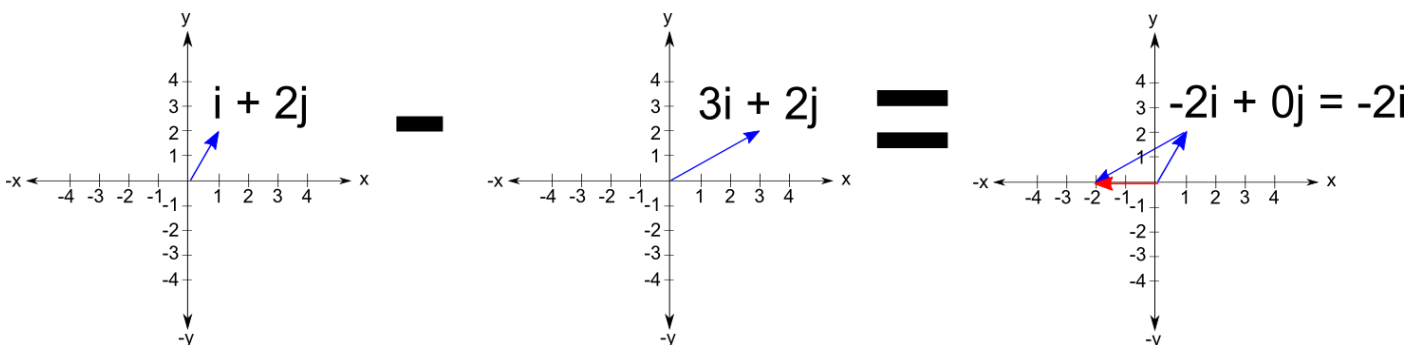
2.1.3 Addition and Subtraction of Vectors

When adding vectors all we need to do is to sum their horizontal components to arrive at the horizontal component for the resultant vector. Likewise, if we add the vertical components of two vectors then we arrive at the vertical component of the resultant vector. Let's see this pictorially...



The drawing shows the **resultant vector** in red. The action of adding vectors produces a resultant. The easy way is to place the vectors in a chain and then the resultant starts at the beginning of the chain and end at the end of the chain (the diagram on the right shows the two blue vectors chained together).

Subtraction of vectors is no more complicated than reversing the direction of the vector which is preceded by the '-' sign. An example...



After reversing the subtracted vector (after all a '-' sign does mean to reverse something) we do the same as vector addition – chain the vectors and draw the resultant to complete the triangle.

2.1.4 Multiplication by a Scalar

It was mentioned at the start of this section that a scalar is just a number – it has no direction. If we wish to multiply a vector by a scalar (i.e. scale the vector – make it bigger or smaller) then we just multiply the vector's magnitude by the scalar, leaving the angle alone.

2.1.5 Graphical Methods

The main graphical method has been covered in previous sections. Just be wary of labelling for axes, as you might see a,b,c or x,y,z or i,j,k or some other system used. Vectors can exist in more than three dimensions – I dare you to draw a 4-D vector! When we discover matrices we will see that 4-D, 5-D and more

dimensions are easily handled with matrix methods (drawings belong to humans but mathematics belongs to the universe).

2.2 Matrix Operations and Vectors

2.2.1 Vectors in Matrix form

Vectors may easily be expressed in terms of matrices (singular term is ‘matrix’, plural is ‘matrices’). Let’s look at a DC network with two loops. A pair of simultaneous equations might result...

$$6i_1 - 2i_2 = 6$$

$$3i_1 + 4i_2 = 18$$

There are two unknowns in these equations, i_1 and i_2 . Our job is to find them. Let’s express this system of equations in matrix form...

$$\begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

The way to read these matrices is as follows...

- Take the 6 and the -2 and then slide them along to the right as a pair. Then rotate 90° clockwise and drop them so that they sit next to i_1 and i_2 . Then multiply out: 6 lots of i_1 plus -2 lots of i_2 . That equals the 6 on the RHS. There’s your first equation formed.
- Next, take the 3 and the 4 and then slide them along to the right as a pair. Then rotate 90° clockwise and drop them so that they sit next to i_1 and i_2 . Then multiply out: 3 lots of i_1 plus 4 lots of i_2 . That equals the 18 on the RHS. There’s your second equation formed.

As you will see later, there are a few ways to solve this matrix expression (in other words, find the two currents) – we can use determinants, matrix inverse or Gaussian elimination.

2.2.2 Square and Rectangular Matrices

A square matrix is, you guessed it, square. Let’s look at one...

$$\text{square matrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 14 & 6 \\ 7 & 0 & 5 \end{bmatrix}$$

You may also guess that a rectangular matrix looks, well, rectangular. Here’s one...

$$\text{rectangular matrix} \begin{bmatrix} 9 & 5 \\ 8 & -4 \\ -5 & 7 \end{bmatrix}$$

Matrix Addition

To be able to add together two matrices we need them to have the same number of rows and the same number of columns (the matrices can be square or rectangular though, not mixed).

Some examples ...

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

The contents of each corresponding cell have simply been added to find the result of the addition. For example, the top left position in the first matrix contains a 1 and the top left position in the second matrix contains a 5. We add that 1 and 5 to produce the resulting 6 at the top left of the solution. Simple.

Here we have used rounded brackets deliberately, to show you that these are often used in textbooks, but are the same as the rectangular brackets previously shown. Another ...

$$\begin{bmatrix} -1 & 2 \\ -8 & 5 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ -5 & 0 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ -13 & 5 \\ 4 & 0 \end{bmatrix}$$

Matrix Subtraction

To be able to subtract one matrix from another we need them to have the same number of rows and the same number of columns (the matrices can be square or rectangular though, not mixed).

Some examples ...

$$\begin{pmatrix} 1 & 6 \\ 8 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 5 & -5 \end{pmatrix}$$

The contents of each corresponding cell have simply been subtracted to find the result of the subtraction. For example, the top left position in the first matrix contains a 1 and the top left position in the second matrix contains a 5. We subtract that 5 from that 1 to produce the resulting -4 at the top left of the solution. Simple.

Another ...

$$\begin{bmatrix} -1 & 2 \\ -8 & 5 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 10 \\ -5 & 0 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ -3 & 5 \\ 0 & 12 \end{bmatrix}$$

Matrix Multiplication

We may multiply a matrix by a scalar (i.e. just a real number, such as 1 or 7 or -6 or 4.35). To do so we simply multiply the contents of that matrix by the scalar. An example ...

$$2 \begin{bmatrix} 3 & 5 & -4 \\ 9 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 10 & -8 \\ 18 & 0 & 4 \end{bmatrix}$$

A more involved multiplication process involves the multiplication of one matrix by another. For this to be valid the number of columns in the first matrix must equal the number of rows in the second matrix, and, the number of rows in the first matrix must equal the number of columns in the second matrix. Let's look at an example and then explain how it is done ...

$$\begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 7 & 2 & 5 \\ 0 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -13 & 0 \\ 14 & -5 & 7 \\ 42 & 0 & 26 \end{bmatrix}$$

Perhaps this looks confusing? This is how it works; the first matrix has a 1 and a 5 in its first row, so we group those two numbers together and rotate them by 90 degrees. Now the 1 is above the 5. We take that rotated grouping and move it next to the first column in the second matrix. Then we multiply and add to find the result for the corresponding cell in the resulting matrix. This will give $(1 \times 7) + (5 \times 0) = 7$. We do the same with the contents of the first row from the first matrix to the contents of the second column in the second matrix i.e. $((1 \times 2) + (5 \times -3) = -13$. We do the same with the contents of the first row from the first matrix to the contents of the third column in the second matrix i.e. $((1 \times 5) + (5 \times -1) = 0$. Now we simply repeat the process using the second row in matrix one, generating the 2nd row in the answer, then use the third row in matrix one to generate the 3rd row in the answer. On now to another topic ...

An important property of square and rectangular matrices is known as the **main diagonal** or **leading diagonal**. The main diagonal consists of the terms sitting on a straight line drawn from the top left to the bottom right. The square matrix above has a main diagonal of 1, 14, 5 and the rectangular matrix has a main diagonal of 9, 7.

A square matrix can be very useful if it contains 1's on the main diagonal and 0's everywhere else, like so...

$$2 \times 2 \text{ identity matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3 \times 3 \text{ identity matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These identity matrices are very useful and will be used later on in this workbook.

2.2.3 Row and Column Vectors

More good guesswork tells us what a row vector looks like...

$$[7 \quad 3 \quad -5]$$

... and the look of a column vector...

$$\begin{bmatrix} 12 \\ 88 \\ -14 \end{bmatrix}$$

2.2.4 Significance of the Determinant

For square matrices (and only square matrices) we may find a quantity known as the **determinant**. If the determinant of a square matrix is non-zero then the system of equations has a unique solution (good). If the determinant has come out to be zero then there is either no solution or many solutions (usually bad).

The next section demonstrates how to evaluate the determinant for a square matrix.

2.2.5 Determinant for a 2 x 2 and 3 x 3 Matrix

The determinant of a square matrix is very useful in finding unknown quantities (like currents, voltages etc.) in systems of equations. Let's see if we can find the determinant for a 2 X 2 matrix...

$$\begin{vmatrix} 6 & -2 \\ 3 & 4 \end{vmatrix} = (6 \times 4) - (-2 \times 3) = 24 + 6 = 30$$

Notice that the determinant has been placed within special symbols $\begin{vmatrix} & \\ & \end{vmatrix}$.

What we do is to multiply the numbers on the leading diagonal (6 X 4) and then subtract the numbers on the other diagonal (-2 X 3). That obviously gives 30. This is non-zero so that tells us that the solution should be unique (i.e. we can easily find the unknown quantities).

A few more examples...

$$\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} = 14$$

$$\begin{vmatrix} -6 & 8 \\ 2 & -4 \end{vmatrix} = 8$$

$$\begin{vmatrix} -6 & 2 \\ 9 & 3 \end{vmatrix} = -36$$

$$\begin{vmatrix} 3 & 18 \\ 2 & 12 \end{vmatrix} = 0$$

Now let's see what happens with the determinant of a 3 x 3 matrix. Each cell (element) in a 3 x 3 matrix has a **minor**, which is found by covering up the corresponding row and column which contains that minor, and then evaluating its determinant. Let's look at an example ...

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The minor of element 4 is obtained by covering up its row and column, like this ...

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now remove those red elements and we have ...

$$\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

Change this matrix into a determinant and that will be the minor of element 4 ...

$$\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = (2 \times 9) - (3 \times 8) = 18 - 24 = -6$$

So, the minor of element 4 is -6.

There is still a bit more to consider. We have a sign associated with each minor, and this is really important. Let's see how this sign 'pattern' works ...

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We see that element 4 lives on row 2 column 1, which requires a '-' sign. We therefore need to write ...

$$-(-6) = 6$$

Once we introduce the sign to a minor we call it the **cofactor**.

Now we are in a position to evaluate the determinant of a 3 x 3 matrix. Let's look at an example ...

$$\begin{vmatrix} 4 & 2 & -1 \\ 6 & -2 & 0 \\ 1 & -4 & 8 \end{vmatrix}$$

We can actually pick any row or column we like to determine the sum of products of any row or column and its cofactors. Why make life hard for ourselves when we see that there is a 0 in column 3, row 2; making those multiplications zero and shortening our work. Here's how it goes ...

$$\begin{aligned} \begin{vmatrix} 4 & 2 & -1 \\ 6 & -2 & 0 \\ 1 & -4 & 8 \end{vmatrix} &= -6 \begin{vmatrix} 2 & -1 \\ -4 & 8 \end{vmatrix} + (-2) \begin{vmatrix} 4 & -1 \\ 1 & 8 \end{vmatrix} + 0 \begin{vmatrix} 4 & 2 \\ 1 & -4 \end{vmatrix} \\ &= -6 \begin{vmatrix} 2 & -1 \\ -4 & 8 \end{vmatrix} + (-2) \begin{vmatrix} 4 & -1 \\ 1 & 8 \end{vmatrix} \\ &= -6((2 \times 8) - (-1 \times -4)) - 2((4 \times 8) - (-1 \times 1)) \\ &= -6(16 - 4) - 2(32 - -1) \\ &= -6(12) - 2(33) \\ &= -72 - 66 = \mathbf{-138} \end{aligned}$$

Please test this out with [this online calculator](#). You will find that it is correct.

2.2.6 The Inverse of a 2x2 Matrix

If a matrix is given the symbol **A** then the *inverse* of that matrix is written **A⁻¹**. Let's begin our analysis by considering a 2 X 2 matrix...

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of a 2 X 2 matrix is then defined as follows...

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Let's look at a quick example of this...

$$A = \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & 2 \\ -3 & 6 \end{bmatrix} = \frac{1}{24 - -6} \begin{bmatrix} 4 & 2 \\ -3 & 6 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 4 & 2 \\ -3 & 6 \end{bmatrix}$$

You might be wondering what use this inverse actually is. Well, a bit earlier on we introduced something called the identity matrix. We shall give the identity matrix the symbol I and re-introduce it here...

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can express our DC current problem as vectors...

$$Ai = n$$

where;

- A is the matrix of coefficients... $\begin{bmatrix} 6 & -2 \\ 3 & 4 \end{bmatrix}$ also considered as a vector
- i is the column vector $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$
- n is the numbers on the RHS in vector form $\begin{bmatrix} 6 \\ 18 \end{bmatrix}$

KEY FACT

When we multiply a matrix by its inverse we get the identity matrix I . Let's try multiplying both sides by the inverse of A and see what happens...

$$A^{-1}Ai = A^{-1}n$$

Since $A^{-1}A = I$ then

$$Ii = A^{-1}n$$

Another useful fact is that in the vector world, multiplying anything by I doesn't change it (just like multiplying anything by 1 never changes it) so we may now write...

$$i = A^{-1}n$$

This last expression (much highlighted) is what we need to solve systems of vector equations. This not only works for 2 x 2 systems but for larger systems also.

Let's now see if we can finally find those two currents...

$$i = A^{-1}n = \frac{1}{30} \begin{bmatrix} 4 & 2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

To resolve the i vector (our answer) we need to slide that 4 and 2 pair over to the right, rotate clockwise by 90° and drop down so that the 4 sits next to the 6 and the 2 sits next to the 18. We then multiply the new pairs and add their results, giving $(4 \times 6) + (2 \times 18) = 24 + 36 = 60$.

We then do the same with the -3 and the 6 pair. Slide, rotate and drop, giving $(-3 \times 6) + (6 \times 18) = -18 + 108 = 90$.

Our solution development now simplifies to...

$$i = A^{-1}n = \frac{1}{30} \begin{bmatrix} 60 \\ 90 \end{bmatrix}$$

We mentioned earlier that multiplying a scalar by a vector simply means multiplying the scalar by the magnitude, so the 60 must be multiplied by 1/30 giving 2 and the 90 must also be multiplied by 1/30 giving 3. Finally, we arrive at our answer...

$$i = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

That's a vector way of saying that current i_1 is 2 Amps and current i_2 is 3 Amps.

2.2.7 Gaussian Elimination for 2 X 2 Systems

This is a quick method to find the unknowns in a set of simultaneous equations. Given two simultaneous equations...

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

NOTE: the first subscript refers to the row number and the second subscript refers to the column number.

We'd like to find those x and y terms, which are just mere numbers. The approach is to try to eliminate the lower left term completely so that we may then easily see the value of y . Once that is known then we use back substitution to find x . This is best explained with an example...

$$6i_1 - 2i_2 = 6$$

$$3i_1 + 4i_2 = 18$$

We'd like to eliminate the lower left term – how can we do that? Well, what if we multiply the top equation by -0.5? That will then give us the pair of equations...

$$-3i_1 + i_2 = -3$$

$$3i_1 + 4i_2 = 18$$

If we now add these equations (like for like, working in geographic columns) we get...

$$0 + 5i_2 = 15$$

$$\therefore 5i_2 = 15 \quad \therefore i_2 = 15/5 = 3$$

Not too much work there to find that i_2 has a value of 3 Amps. Now we can pick one of the original equations (how about the first one?) to find a value for i_1 ...

$$6i_1 - 2i_2 = 6 \quad \therefore 6i_1 - 2(3) = 6 \quad \therefore 6i_1 = 12 \quad \therefore i_1 = 12/6 = 2$$

That little bit more work gave us a value for i_1 of 2 Amps.