Pearson BTEC Levels 5 Higher Nationals in Engineering (RQF)

Unit 39: Further Mathematics

Unit Workbook 3

in a series of 4 for this unit

Learning Outcome 3

Graphical & Numerical Methods

3.1 Graphical Techniques

3.1.1 Curve Sketching Strategies

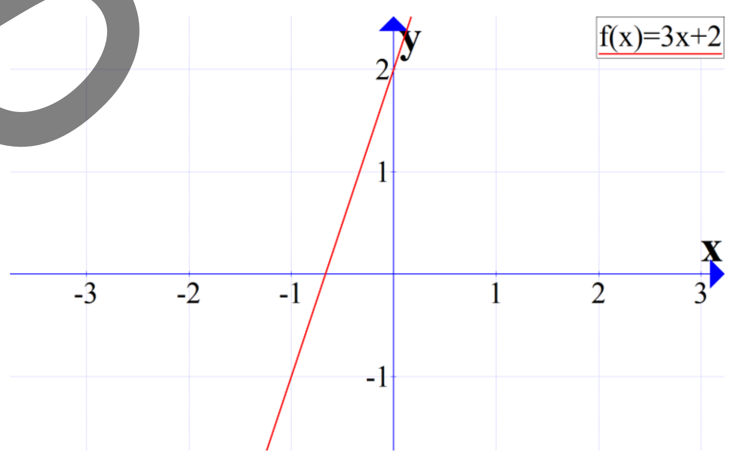
When examining the expression for a function it is usually a good idea to be able to sketch the function roughly. There are a number of ways to go about this, as follows;

- Let the independent variable (usually x or t) be zero and see what value that gives for the dependent variable (usually y or voltage or current). This will give the point where the function crosses the vertical axis. Always a good starting point in curve sketching.
- Let the dependent variable be zero and possibly then determine the roots of the equation – i.e. at which point(s) does it cross the horizontal axis?
- If the highest power of the independent variable is 1 then the function is a straight line.
- Check if it is an EVEN function i.e. does $f(x) = f(-x)$? If this is the case then the function is symmetrical about the vertical axis. A common example is $y = \cos(x)$.
- Check if it is an ODD function i.e. does $f(-x) = -f(x)$? If this is the case then the function is symmetrical about the origin. A common example is $y = \sin(x)$.
- Differentiate the function to determine any turning point. If possible, differentiate again to determine whether the turning point is a maxima or minima (peak/trough) by checking the sign on the result. Negative sign means a maxima, positive sign means a minima.
- A quadratic equation (i.e. where the highest power of the independent variable is 2) will have one turning point. A parabola will be produced.
- Determine any asymptotes (explained in a while).

3.1.2 Curve Sketching Examples

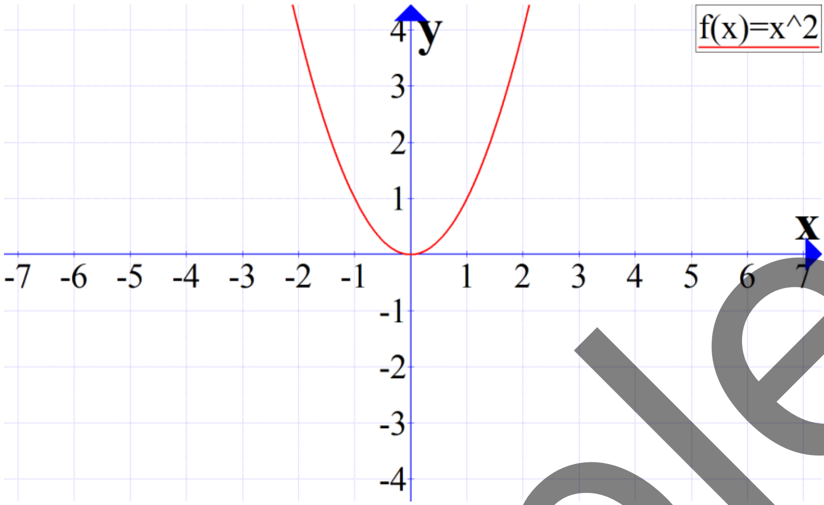
Note: In these examples Graph software is used to generate the plots. ***Your curve sketches will be drawn by hand for the assignment***.

Linear function



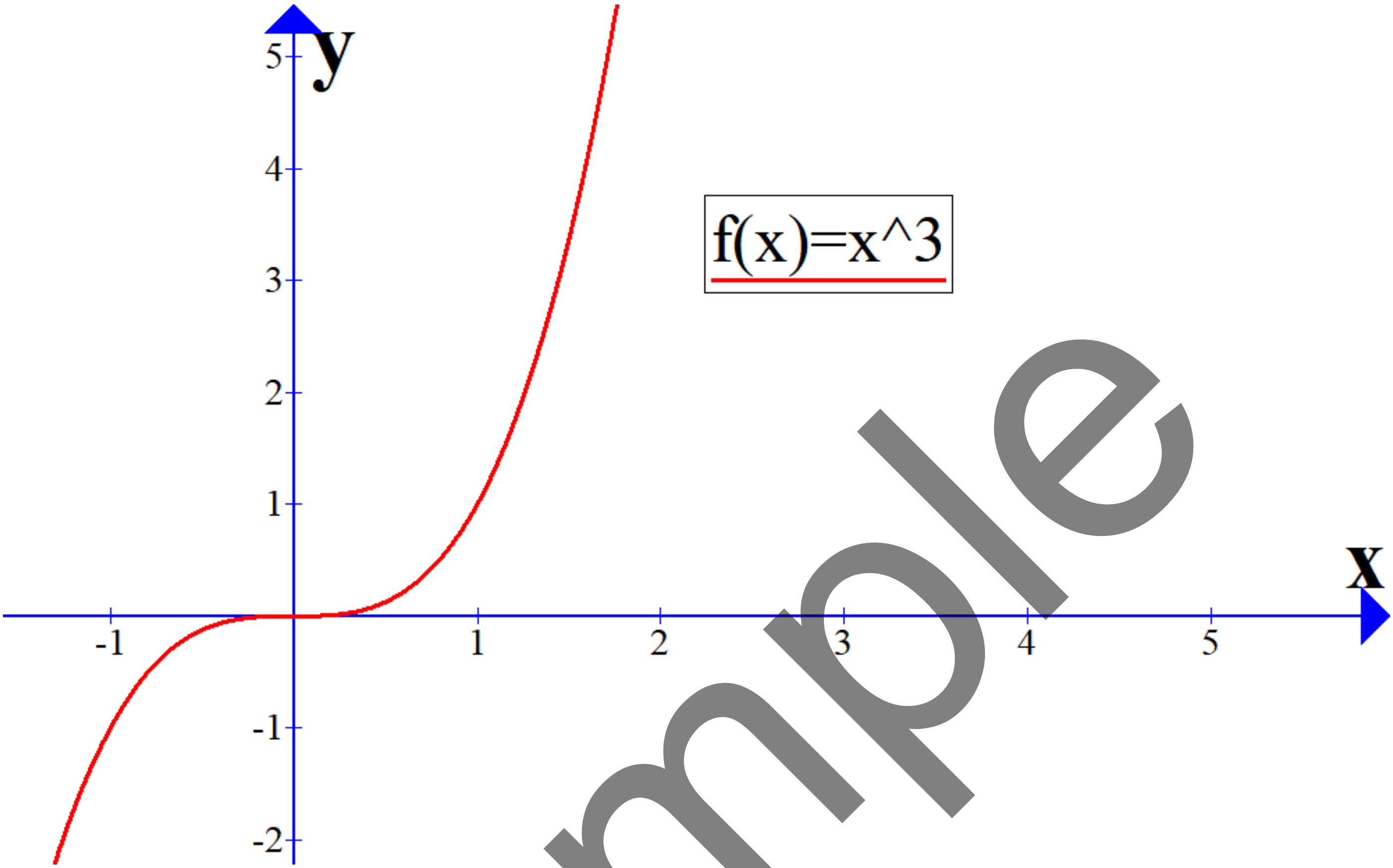
- Letting $x = 0$ makes $y = 2$ which is where the function crosses the y axis
- Letting $y = 0$ gives $3x + 2 = 0 \quad \therefore \quad x = -2/3$ which is where the function crosses the x axis
- The slope of the function is the coefficient of x, which is 3

Quadratic Function



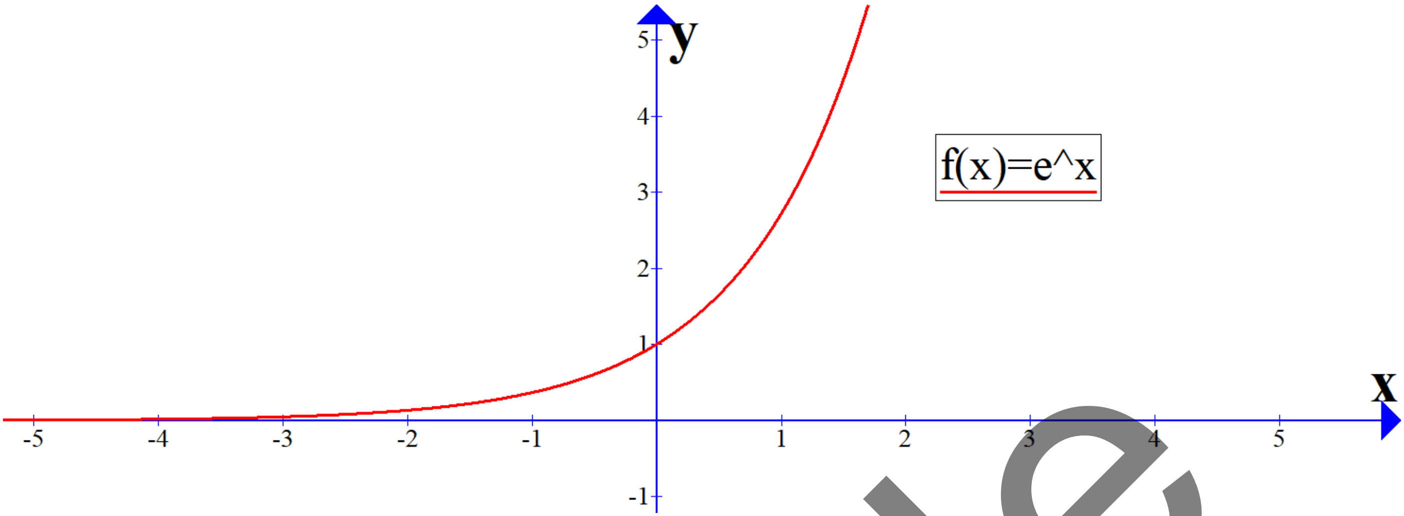
- Since $y = x^2$ (raising to a power in Graph needs the ^ symbol, as indicated on the legend) then letting $x = 0$ gives $y = 0$. So, the curve meets the axes at (0, 0) – i.e. the origin.
- Since the function is a quadratic (highest power is 2) then there is one turning point. To find out what type it is we find the sign of the 2nd differential. Differentiating once gives $y' = 2x$ and differentiating again gives $y'' = 2$. The sign here is positive, indicating a local minima (as indicated in section 2.1.1).
- Also, since the function is quadratic, we have a parabola.

Cubic Function



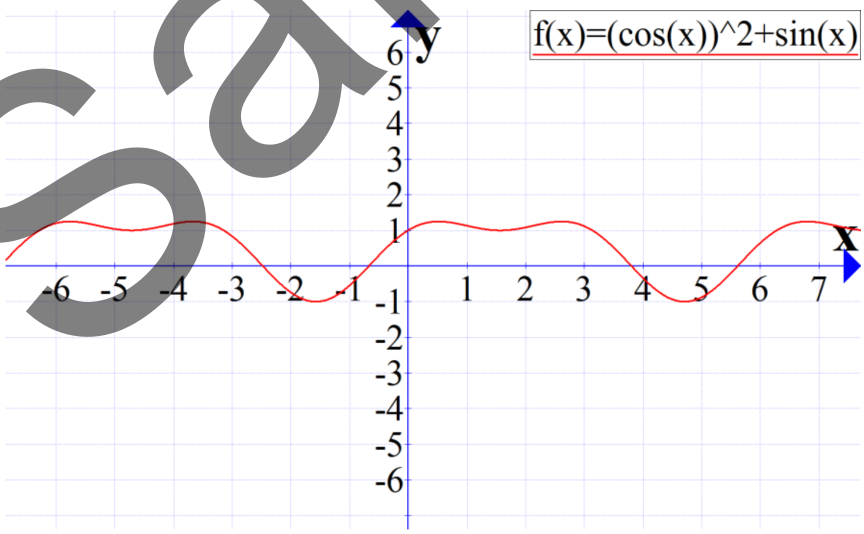
- Since $y = x^3$ then letting $x = 0$ gives $y = 0$. So, the curve meets the axes at (0, 0) – i.e. the origin.
- The origin is a turning point for this function.
- Since the function is cubic (highest power is 3) then there is at least one turning point. To find out what type it is we find the sign of the 2nd differential. Differentiating once gives $y' = 3x^2$ and differentiating again gives $y'' = 6x$. The sign here is positive, indicating a local minima.

Exponential Function



- Since $y = e^x$ then letting $x = 0$ gives $y = 1$. So, the curve meets the vertical axis at (0, 1).
- Since the function is exponential then differentiating once gives $y' = e^x$ and differentiating again gives $y'' = e^x$. In other words, the differential of this function is the function itself.
- Letting y = 0 will imply that x will be 'minus infinity'.

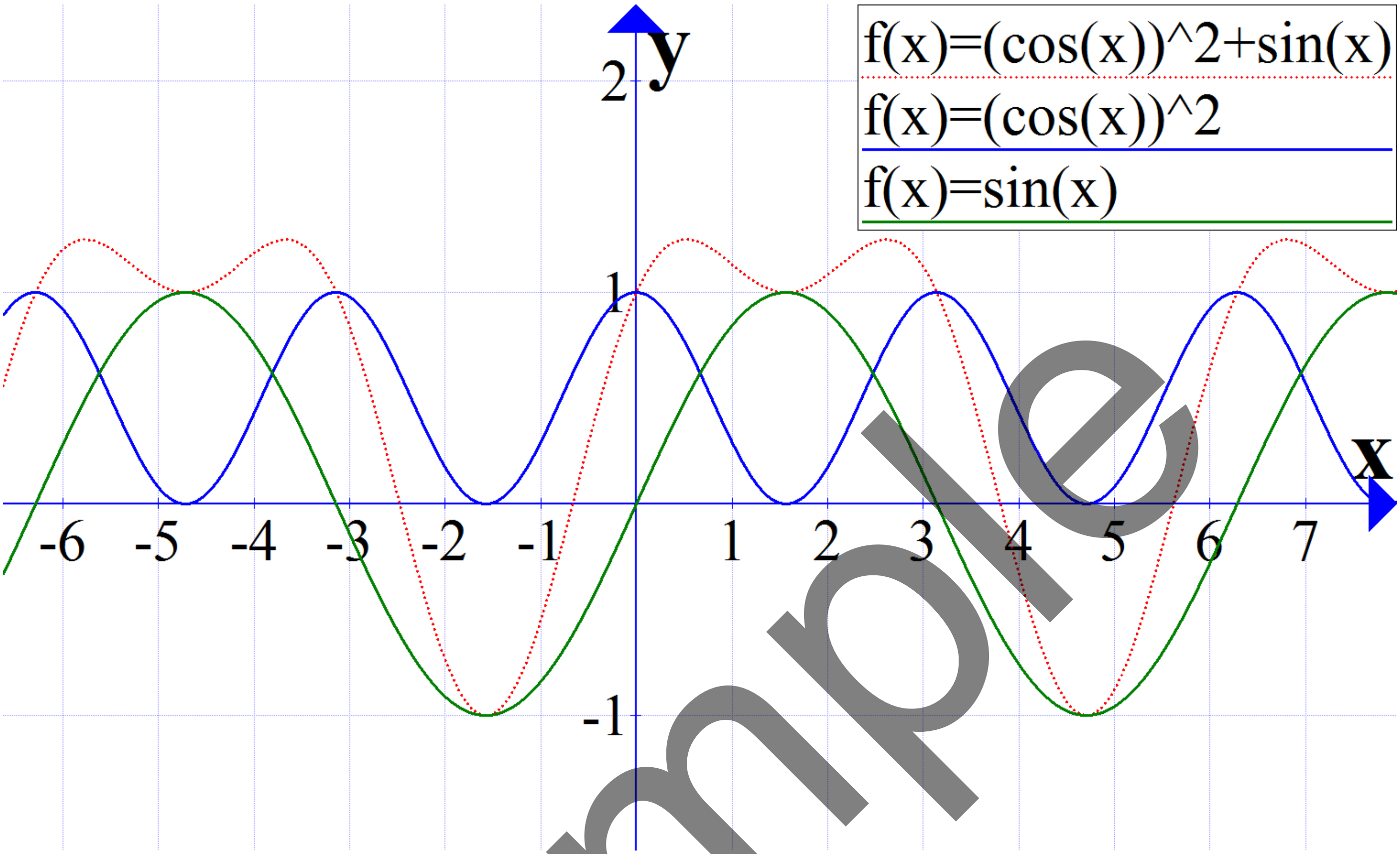
Mixed Trigonometric Function



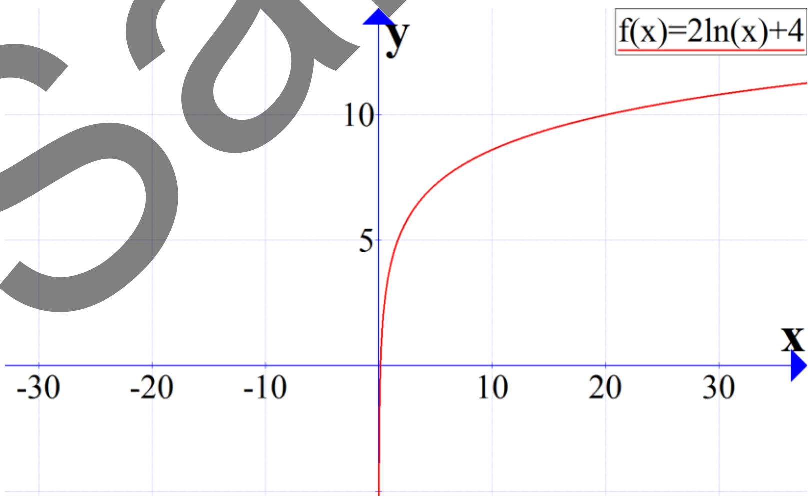
Here, our function is $y = cos^2(x) + \sin(x)$. Letting $x = 0$ gives $y = 1$, which is where the function hits the y axis.

Letting $y = 0$ gives $cos^2(x) = -\sin(x)$. This is rather difficult to tackle, since we are only sketching and the process should be fairly quick.

The best way to approach this sketch problem is to make a sketch of $cos^2(x)$ and another sketch of $\sin(x)$ then ***ADD the waveforms together***, like so…

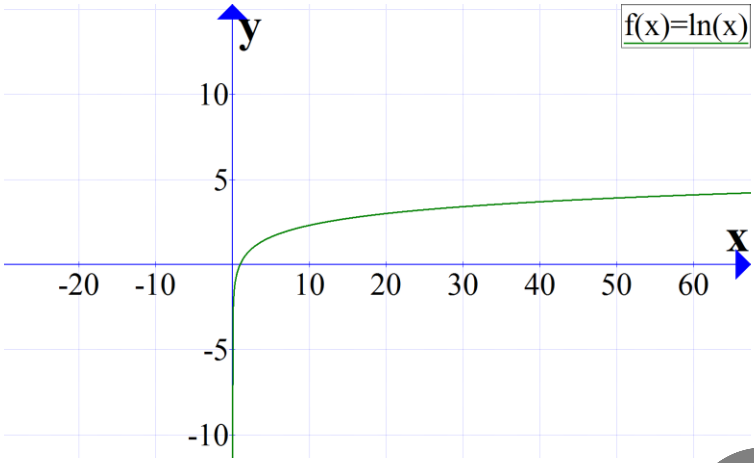


Logarithmic Function

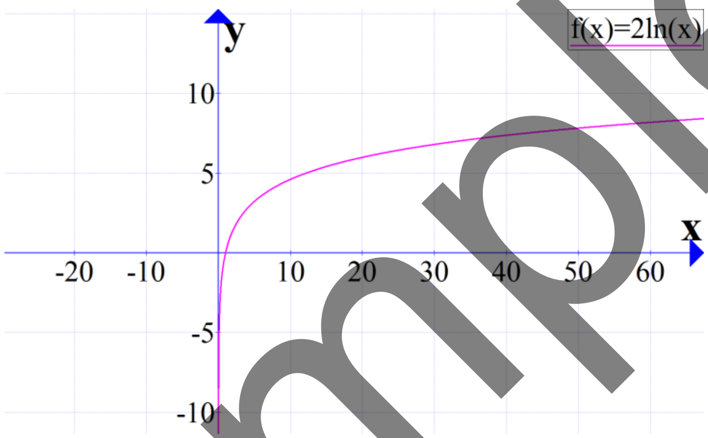


This function is $y = 2log_n(x) + 4$ and we cannot make $x = 0$ here as it will give minus infinity (a calculator will give an error of you ask for the log of zero, try it) which looks evident from the plot above. The way to go about this problem is to draw $y = log_n(x)$…

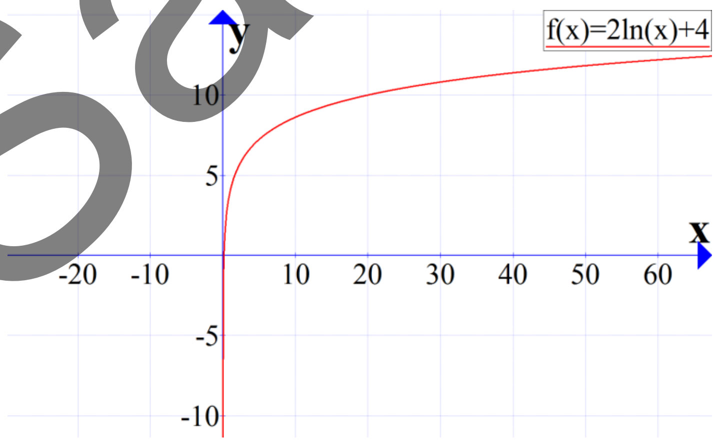
edexcel Approved Centre



…then multiply by 2…



…then add 4…



What we notice when zooming into the green function is that the curve crosses the x axis at a point greater than zero. Let's find this point…

$$y = log_n(x) \quad \therefore \quad 0 = log_n(x) \quad \therefore \quad x = e^0 = 1$$

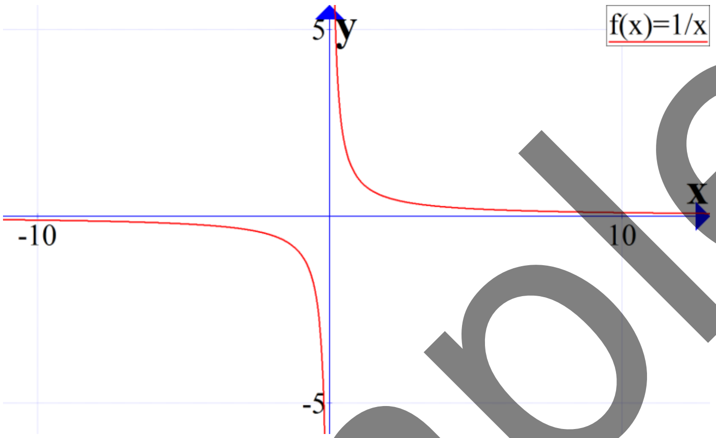
So, the function crosses the x axis at 1.

When attempting assignment questions you must sketch your curves by hand. There is no problem in using Graph (or other free simulators) to check that you're on the right lines.

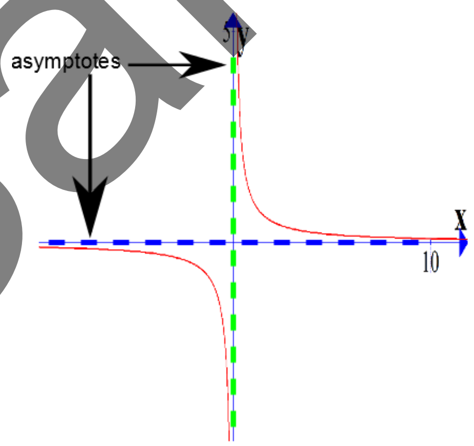
3.1.3 Asymptotes

As a curve approaches infinity we can draw a tangent to this curve at infinity (a good guess at infinity anyway). The line which we draw is known as an asymptote.

Here's a graph of $y = 1/x$…



Clearly the graph goes to plus infinity as we travel up the positive vertical axis. It goes to negative infinity as we travel down the negative vertical axis. We can then draw a straight line between these two points, which will be an asymptote. We can do the same for another obvious asymptote along the horizontal. The two asymptotes are shown below…



… in thick green and blue dashed lines.

The equation for the green asymptote appears to be x = 0 and the blue one appears to be y = 0. Let's see how we determine these mathematically…

To find asymptotes parallel to the x axis we equate coefficients of the highest power of x to zero…

$$y = 1/x \quad \therefore \quad yx = 0$$

edexcel Approved Centre

The highest power of x here is 1 and that coefficient is y, so we say that there is an asymptote parallel to the x axis given by $y = 0$. That agrees with what we observe.

To find asymptotes parallel to the y axis we equate coefficients of the highest power of y to zero…

$$y = 1/x \quad \therefore \quad xy = 0$$

The highest power of y here is 1 and that coefficient is x, so we say that there is an asymptote parallel to the y axis given by $x = 0$. That also agrees with what we observe.

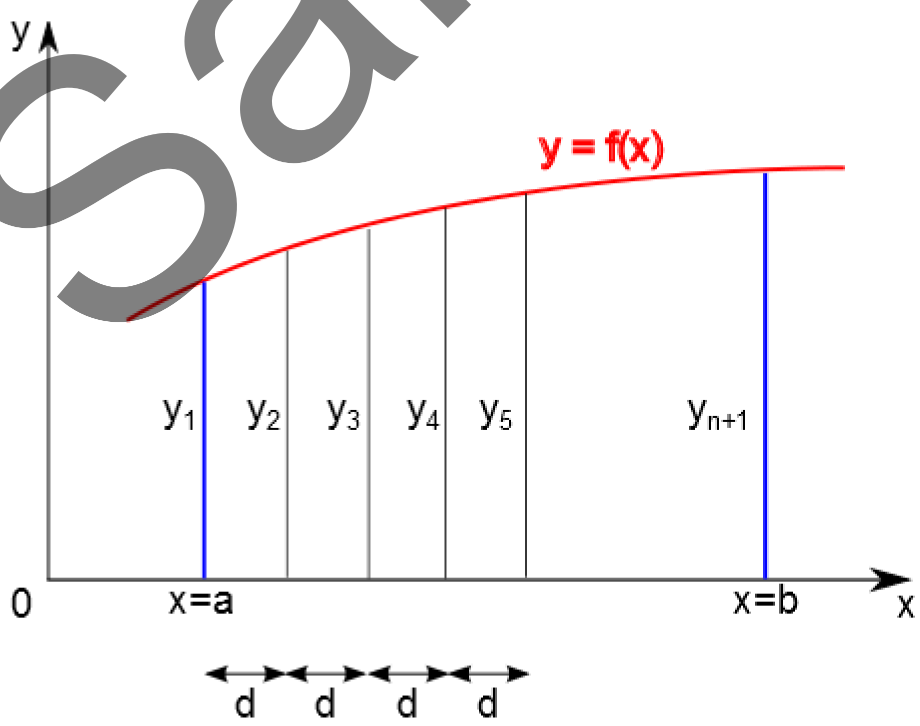
3.2 Numerical Integration

Sometimes it is difficult to use analytical methods to determine the answer to definite integrals, which require the area under a curve bounded by the horizontal axis and two limits to be calculated. In this section you will be introduced to a very useful numerical method to find such solutions.

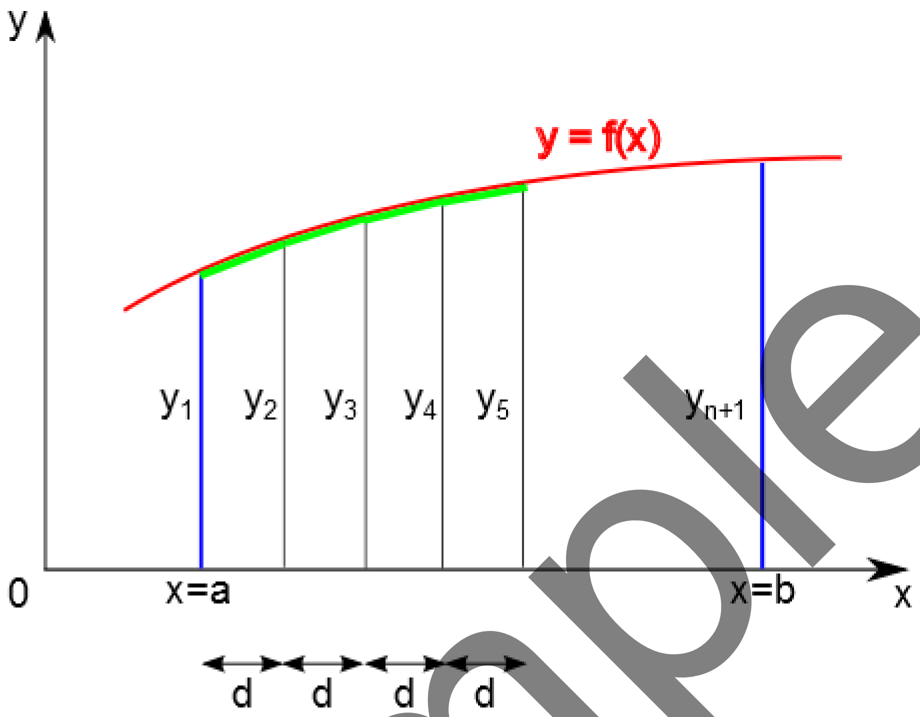
3.2.1 The Trapezoidal Rule

With reference to the diagram below, what we would like to do is estimate the area under the curve bounded by the curve itself, the blue limits and the x axis.

We need to divide the area under the curve into n equally wide sections (of width d) and we label these ordinates as $y_1, y_2, y_3 \ldots\ y_{n+1}$, as shown.

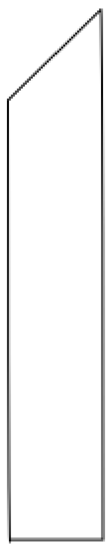


If we now join together the tops of each of the ordinates with straight lines we shall have n trapezoids, as below…



These added straight lines are shown in green. What remains to do then is to add up all of the areas for the trapezoids.

Remember how to work out the area of a trapezoid…



Area = 0.5 x (sum of parallel lengths) X width

We can then write…