Pearson BTEC Levels 5 Higher Nationals in Engineering (RQF)

Unit 39: Further Mathematics

Unit Workbook 4

in a series of 4 for this unit

Learning Outcome 4

Ordinary Differential Equations

4.1 First Order Differential Equations

[Your solutions may be checked with this online calculator]

4.1.1 Separation of Variables

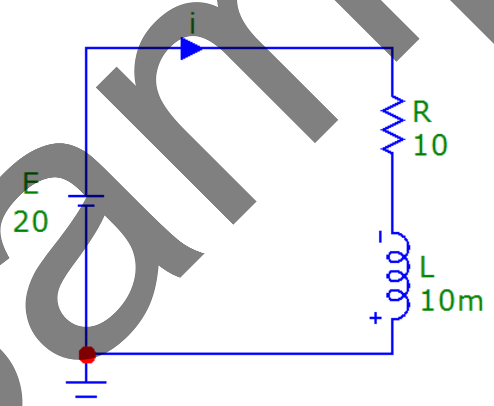
First Order differential equations involve a single derivative (such as $\frac{dy}{dx}$). Consider first order differential equations of the form…

$$\frac{dy}{dx} = f(x) \ \ or \ \ \frac{dy}{dx} = f(y) \ \ or \ \ \frac{dy}{dx} = f(x) \cdot f(y)$$

Each of these may be solved by direct integration after the separation of variables. By doing so we place the $y's$ on one side and the $x's$ on the other side. Series RL and series RC circuit analysis yields first order differential equations which may be treated in this way. Let's examine these circuits to see if we can first of all produce a differential equation for each of them, and then try to solve the RL circuit by separation of variables.

Worked Example 1

Analyse the RL circuit below and formulate a mathematical model which represents the circuit behaviour by a first order differential equation.



We need to make use of Kirchhoff's Voltage Law, which states that the voltages in a closed loop sum to zero. We note the voltages across the resistor and inductor as…

$$V_R = iR$$

$$V_L = L\frac{di}{dt}$$

So we may now write…

$$E = V_R + V_L$$

$$\therefore \quad E = iR + L\frac{di}{dt}$$

We always try to make the coefficient of the differential 1 so dividing through by L will achieve this…

$$\frac{E}{L} = \frac{iR}{L} + \frac{L}{L} \cdot \frac{di}{dt}$$

Tidying this up and re-arranging gives…

$$\frac{di}{dt} + \left(\frac{R}{L}\right) i = \frac{E}{L}$$

All we need to do now is to insert those component values…

$$\frac{di}{dt} + \left(\frac{10}{0.01}\right) i = \frac{20}{0.01}$$

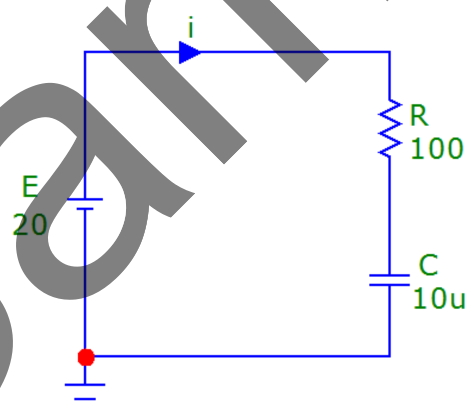
Which allows us now to write the final differential equation to represent this circuit…

$$\boldsymbol{\frac{di}{dt} + 1000i = 2000}$$

Let's now perform the same analysis on an RC circuit…

Worked Example 2

Analyse the RC circuit below and formulate a mathematical model which represents the circuit behaviour by a first order differential equation.



We again make use of Kirchhoff's Voltage Law, which states that the voltages in a closed loop sum to zero. We note the voltages across the resistor and capacitor as…

$$V_R = iR$$

$$V_C = \frac{1}{C} \int i \, dt$$

So we may now write…

$$E = V_R + V_C$$

$$\therefore \quad E = iR + \frac{1}{C}\int i\,dt$$

Multiply throughout by C…

$$EC = RCi + \int i\,dt$$

Then differentiate each term with respect to t…

$$0 = RC\frac{di}{dt} + i$$

Divide throughout by RC and tidy…

$$\frac{di}{dt} + \left(\frac{1}{RC}\right)i = 0$$

All we need to do now is to insert those component values…

$$\frac{di}{dt} + \left(\frac{1}{100 \times 10 \times 10^{-6}}\right)i = 0$$

Which allows us now to write the final differential equation to represent this circuit…

$$\boldsymbol{\frac{di}{dt} + 1000i = 0}$$

Worked Example 3

Produce an analytical solution for the first order differential equation developed in Worked Example 1 to find an expression for the current. Assume $i = 0$ when $t = 0$.

We had…

$$\frac{di}{dt} + 1000i = 2000$$

We start by transposing…

$$\frac{di}{dt} = 2000 - 1000i$$

Separate the variables (the variables are i and t)…

$$dt = \frac{di}{2000 - 1000i}$$

Integrate both sides with respect to t…

$$\int dt = t = \int \frac{di}{2000 - 1000i}$$

$$\therefore \quad t = \int \frac{di}{2000 - 1000i}$$

$$\therefore \quad t = \int \frac{1}{2000 - 1000i} di$$

Let $u = 2000 - 1000i \quad \therefore \quad \frac{du}{di} = -1000 \quad \therefore \quad \frac{di}{du} = \frac{-1}{1000}$

$$\therefore \quad t = \int \frac{1}{u} di = \int \frac{1}{u}\left(\frac{di}{du}\right) du = \int \frac{1}{u}\left(\frac{-1}{1000}\right) du = \frac{-1}{1000}\int \frac{1}{u} du$$

Remembering that $\int \frac{1}{u} du = log_e(u) + K$ we may write…

$$t = \frac{-1}{1000}[log_e(u) + K]$$

Notice that K is a constant of integration. Mathematicians use C but engineers use K (since we reserve C for capacitance – avoid confusion)

$$\therefore \quad t = \frac{-1}{1000}[log_e(2000 - 1000i) + K]$$

Now we must find the value for the constant of integration. To do this we must be given some initial conditions for the circuit. Fortunately, we were told that $i = 0$ when $t = 0$ so we may find K by putting these values in…

$$0 = \frac{-1}{1000}[log_e(2000 - 0) + K]$$

$$\therefore \quad 0 = \frac{-log_e(2000)}{1000} - \frac{K}{1000}$$

$$\therefore \quad \frac{K}{1000} = \frac{-log_e(2000)}{1000}$$

$$\therefore \quad K = -log_e(2000) = -7.6$$

Having found K we can get back to our solution development (we left it at the blue equation above)…

$$\therefore \quad t = \frac{-1}{1000}[log_e(2000 - 1000i) - 7.6]$$

Transpose…

$$log_e(2000 - 1000i) = 7.6 - 1000t$$

Almost there. Now take the exponential of BOTH sides to get rid of that log term…

$$e^{log_e(2000-1000i)} = 2000 - 1000i = e^{7.6-1000t}$$

Now it looks easy to isolate the current i…

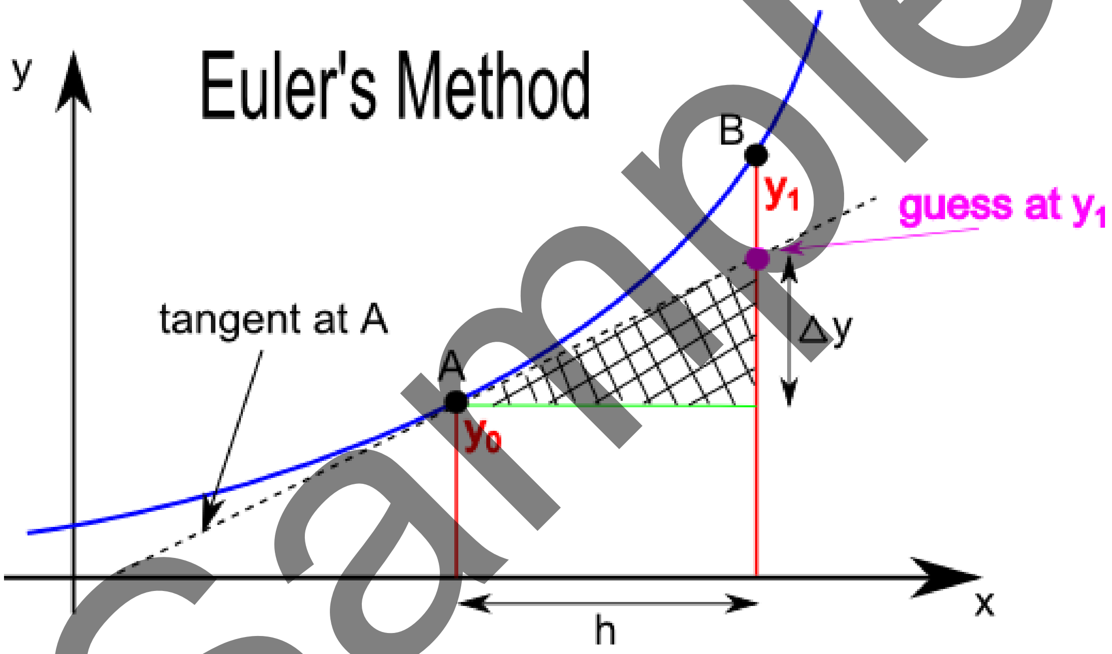
$$1000i = 2000 - e^{7.6-1000t}$$

Dividing both sides by 1000 will get us to our target…

$$\boldsymbol{i = \frac{2000 - e^{7.6-1000t}}{1000}}$$

4.1.2 Euler's Method

We may also solve first order differential equations by Euler's Method, which is a numerical solution, rather than an analytical one. To see what this method is all about take a look at the drawing below…



We have a section of a function to analyse (in blue). If we have a first order differential equation for the function and are given a starting point on the function (A) which has a height on the vertical axis of y_0 then we may make a guess as to the height of another point (b) by drawing a tangent through A and adding the height of the resulting triangle (hashed).

Put quite simply, our best guess at y_1 is to add the length Δy to y_0.

The slope of the hashed triangle is, of course, the same as the slope of the tangent. Since the slope of the tangent is the differential of the function at A then we may represent this as $(y')_0$. The slope of the triangle is given by…

$$slope\ of\ triangle = \frac{\Delta y}{h} \quad \therefore \quad \Delta y = h \times slope\ of\ triangle \quad \therefore \quad \Delta y = h(y')_0$$

So then, we may say that our best guess at y_1 is given by…

$$\boldsymbol{y_1 = y_0 + h(y')_0}$$

This last equation represents Euler's Method for finding the height of a subsequent point on a function. Let's look at an example…

Worked Example 4

Let's re-examine the differential equation which we produced for the RC circuit in Worked Example 2…

$$\frac{di}{dt} + 1000i = 0$$

To use Euler's Method on this we need the differential on its own, so we transpose…

$$\frac{di}{dt} = -1000i$$

To use Euler's Method we need to change $y's$ into $i's$…

$$i_1 = i_0 + h(i')_0$$

Now we need some initial conditions and a point further on in time (h) to analyse. **Assume that when $\boldsymbol{t = 0.001s}$ the current is 0.0736A. We are asked to find the current when time is $\boldsymbol{0.0012s}$.**

Let's write Euler's Method once more so as to realise which bits of information we need…

$$i_1 = i_0 + h(i')_0$$

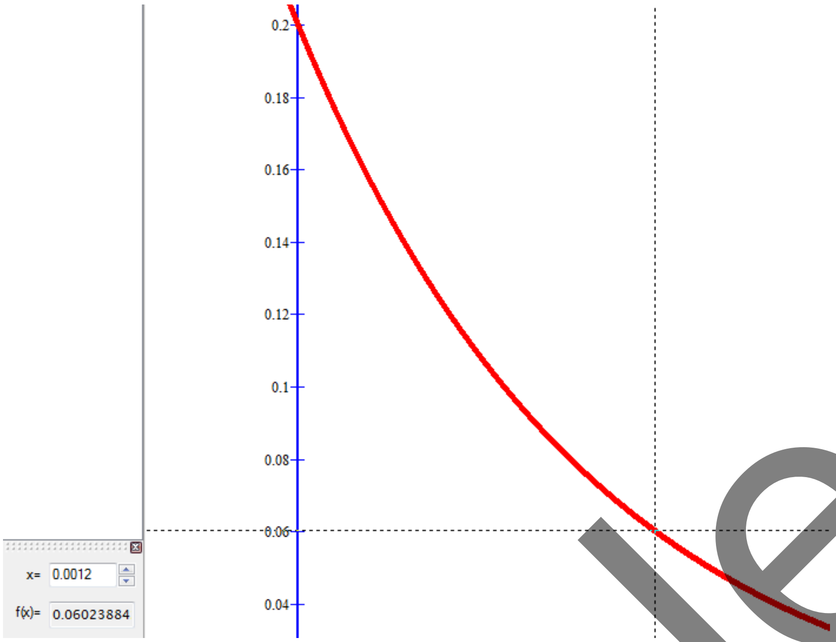
We need:

- i_0, which is the current given in the question i.e. 0.0736A
- h, which is the time difference, i.e. 0.0012s -0.001s = 0.0002s
- $(i')_0$, which is found by checking the differential equation and resolving: -1000(0.0736) = -73.6

We may now find i_1…

$$i_1 = i_0 + h(i')_0 = 0.0736 + 0.0002(-73.6) = 0.0588A$$

This is confirmed as a good guess by checking the result for the circuit in 'Graph'…



Close enough.

4.2 Second Order Differential Equations

There are two types of 2nd order differential equation we shall examine. The first case is…

$$HOMOGENEOUS\ (RHS\ is\ zero):\ \ a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

… and the second case is…

$$NON-HOMOGENEOUS\ (RHS\ is\ non-zero):\ \ a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

4.2.1 Homogeneous 2nd Order Differential Equations

We start with the homogeneous format…

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

To solve such ODE's we deploy an AUXILIARY EQUATION which helps our solution along. This auxiliary equation contains m representations of the differentials within the ODE…

$$m^2\equiv\frac{d^2y}{dx^2}$$

$$m\equiv\frac{dy}{dx}$$

This allows us to write the ODE as…

$$am^2 + bm + c = 0 \quad : AUXILIARY\ EQUATION$$

We then factorise this auxiliary equation and find the roots, which we call m_1 and m_2. The procedure is to then examine these roots to see whether they are…

Real and different: The solution is $y = Ae^{m_1 x} + Be^{m_2 x}$

Real and equal: The solution is $y = (Ax + B)e^{mx}$

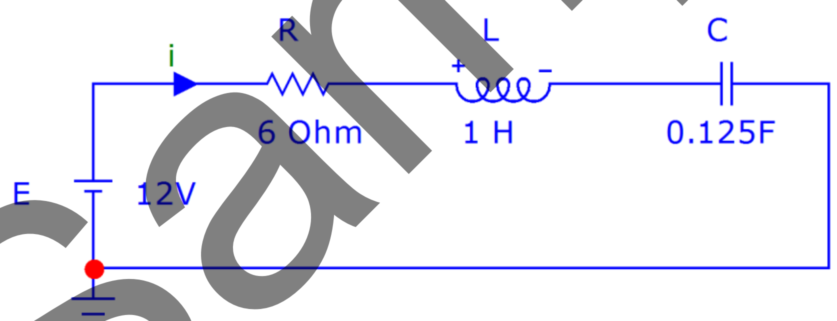
Complex: The solution is $y = e^{m_1 x}\{A\cos(m_2 x) + B\sin(m_2 x)\}$

If we are given some initial conditions in a problem/question then we plug these into our solution to find a more ***particular solution***.

That all looks quite complicated when seen for the first time. Let's develop a 2nd order ODE for an RLC circuit connected to a DC source, by way of a worked example. We'll then use another worked example to see how we can solve such an ODE using the above procedure.

Worked Example 5

Consider the RLC combination below, connected to a DC source



Analyse the circuit and formulate a mathematical model for it by developing a second order homogeneous differential equation involving current and time.

Our starting point is Kirchhoff's Voltage Law (KVL) which says that the sum of voltages in a closed loop is zero. Let's give an individual equation to each of these component voltages…

$$V_R = iR$$

$$V_L = L\frac{di}{dt}$$

$$V_C = \frac{1}{C}\int i\, dt$$

KVL says that $V_R + V_L + V_C = E$ so we may now write…

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E$$

That's not yet and ODE. We need to get rid of that integral. We do so by differentiating every term (which will obliterate the integral)…

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

The E dropped away because it is DC and is a flat voltage (zero slope).

Now we just re-arrange the terms…

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

Finally, put in those component values…

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + \frac{i}{0.125} = 0$$

Tidying…

$$\boldsymbol{\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 0}$$

Worked Example 6

Find a particular solution to Worked Example 5, given that when $t = 0, i = 0$ and ${di}/{dt} = 12$.

We have…

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 0$$

We firstly form the auxiliary equation…

$$m^2 + 6m + 8 = 0$$

Factorise this to find the roots…

$$(m + 2)(m + 4) = 0$$

$$\therefore \quad m_1 = -2 \quad and \quad m_2 = -4$$

These roots are real and different so we use the solution (given earlier)…

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

We are not using x on the horizontal, we are using t. We're not using y either, we're using i, so change this equation to…

$$i = Ae^{m_1 t} + Be^{m_2 t}$$

Put the m's in…

$$i = Ae^{-2t} + Be^{-4t}$$

If we were not given any initial conditions then we would need to stop there. However, we *are* given some initial conditions in this question… $when\ t = 0, i = 0\ \ and\ {}^{di}/_{dt} = 12$

Using $t = 0, i = 0$:

$$i = Ae^{m_1 t} + Be^{m_2 t} = 0 = A(0) + B(0)$$

$$\therefore \quad A + B = 0 \qquad [1]$$

Using ${}^{di}/_{dt} = 12$ when $t = 0$ and $i = 0$:

$$i = Ae^{-2t} + Be^{-4t} \quad \therefore \quad \frac{di}{dt} = -2Ae^{-2t} - 4Be^{-4t} = 12$$

Since $t = 0$ in this last expression then…

$$-2A - 4B = 12 \qquad [2]$$

Now, equations [1] and [2] form a pair of simultaneous equations…

$$\therefore \quad A + B = 0 \qquad [1]$$

$$-2A - 4B = 12 \qquad [2]$$

If we multiply [1] by 2 then we will have…

$$2A + 2B = 0 \qquad [3]$$

Then, ADDING [2] and [3] gives…

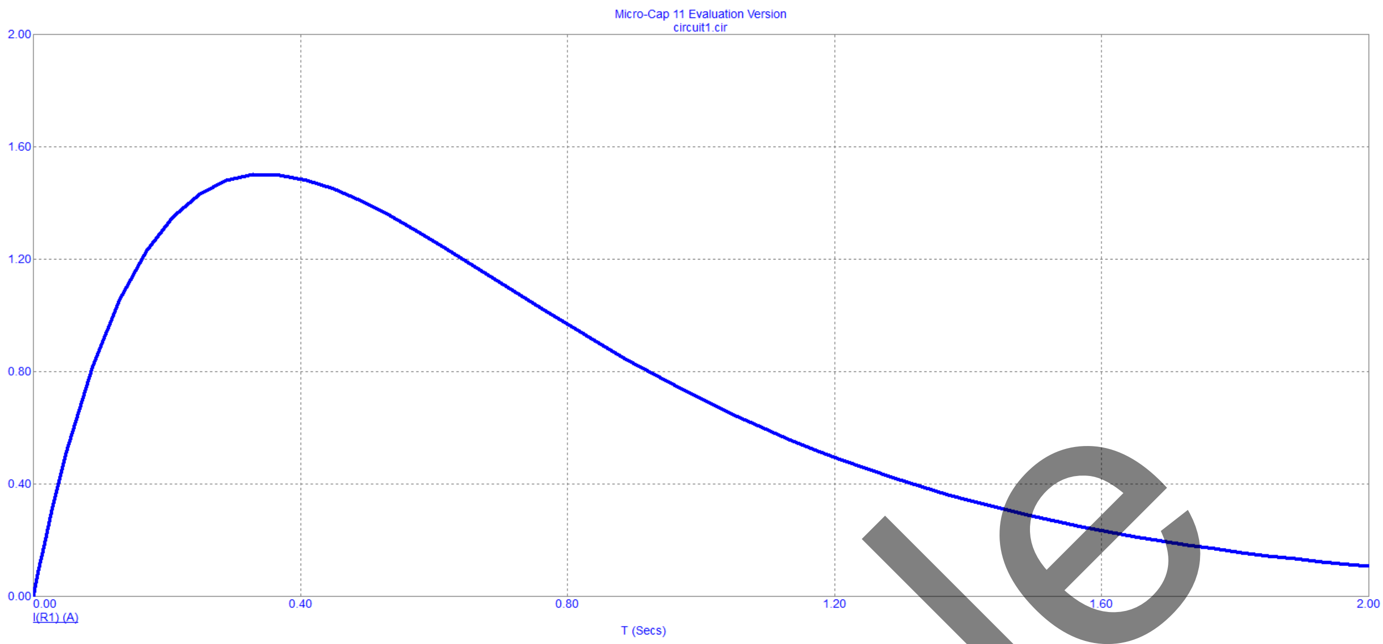
$$-2B = 12 \quad \therefore \quad B = -6$$

Also, since $A + B = 0 \quad \therefore \quad A = -B = - - 6 = 6$

So we have $A = 6$ and $B = -6$, which we then plug into our solution to find the particular solution…

$$\boldsymbol{i = 6e^{-2t} - 6e^{-4t}}$$

This particular solution has the form below…



4.2.2 Non-Homogeneous 2nd Order Differential Equations

As a reminder, these take the general form…

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

The RHS is *not* zero here, it is a function of x. Let's see how we solve these.

The ***general solution*** (GS) is…

$$General\ Solution = Complementary\ Function + Particular\ Integral$$

Or, in shorthand…

$$GS = CF + PI$$

The CF is found by assuming that the RHS is zero and therefore we treat this part of the problem as if it were a homogenous ODE. We are told above that we must also add the PI to this to get the overall solution.

To find the PI we must look at the nature of $f(x)$ on the RHS – is it exponential, sinusoidal, polynomial?

We use a table to find the best PI, as follows…

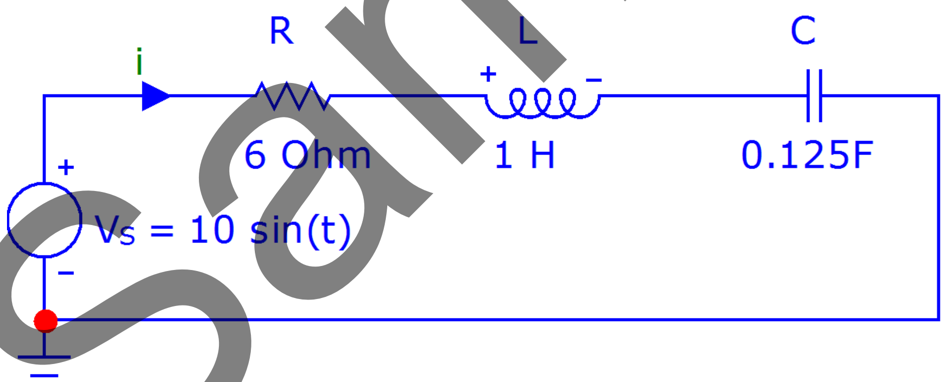
$f(x) =$	Assume $y =$
k	C
kx	$Cx + D$
kx^2	$Cx^2 + Dx + E$
$k\sin(x)$ or $k\cos(x)$	$C\cos(x) + D\sin(x)$
$k\sinh(x)$ or $k\cosh(x)$	$C\cosh(x) + D\sinh(x)$
e^{kx}	Ce^{kx}

Whichever form is most suitable from the right-hand column we select that and differentiate twice. These differentials are then substituted into the original ODE and the letters C and D are evaluated, usually by equating coefficients. We then have the particular integral, which we add to the complementary function to find the general solution. Sounds complicated? It is the first time and then it becomes more apparent what is going on.

Let's develop a 2nd order ODE for an RLC circuit connected to an AC source, by way of a worked example. We'll then use another worked example to see how we can solve such an ODE using the above procedure.

Worked Example 7

Consider the RLC combination below, connected to an AC source



Analyse the circuit and formulate a mathematical model for it by developing a second order non-homogeneous differential equation involving current and time.

Let's produce the ODE…

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 10\sin(t)$$

Differentiating throughout (to get rid of the integral)…

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 10\cos(t)$$

Put the component values in and re-arrange…

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + \frac{i}{0.125} = 10\cos(t)$$

$$\therefore \quad \boldsymbol{\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 10\cos(t)}$$

That's it!

Worked Example 8

Find a general solution (complementary function + particular integral) to the non-homogeneous problem given in Worked Example 7.

In Worked Example 7 we found the ODE…

$$\boldsymbol{\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 10\cos(t)}$$

We are trying to find the general solution (GS), which is given by…

$$GS = CF + PI$$

Find the Complementary Function:

$$\therefore \quad \frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 0$$

We forced it to be zero, which is a requirement when resolving the CF.

We firstly form the auxiliary equation…

$$m^2 + 6m + 8 = 0$$

Factorise this to find the roots…

$$(m+2)(m+4) = 0$$

$$\therefore \quad m_1 = -2 \quad and \quad m_2 = -4$$

$$\therefore \quad \boldsymbol{i = Ae^{-2t} + Be^{-4t}}$$

That's our CF. You will remember that this was already derived in Worked Example 6, but is repeated here for clarity.

Find the Particular Integral:

We see that we have a cosine term on the RHS, so we consult the table and *assume*…

$$i = C\cos(t) + D\sin(t)$$

From this we differentiate twice…

$$\frac{di}{dt} = -C\ sin(t) + D\ cos(t)$$

$$\frac{d^2i}{dt^2} = -C\ cos(t) - D\ sin(t)$$

We now place these last three terms into the circuit ODE…

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 8i = 10\cos(t)$$

$$[-C\ cos(t) - D\ sin(t)] + 6[-C\ sin(t) + D\ cos(t)] + 8[C\cos(t) + D\sin(t)] = 10\cos(t)$$

Multiplying out and re-arranging…

$$[-C + 6D + 8C]\cos(t) + [-D - 6C + 8D]\sin(t) = 10\cos(t)$$

Bring the RHS over to the left…

$$[-C + 6D + 8C - 10]\cos(t) + [-D - 6C + 8D]\sin(t) = 0$$

Add like terms…

$$[7C + 6D - 10]\cos(t) + [-6C + 7D]\sin(t) = 0$$

Now, equating coefficients of cos(t)…

$$7C + 6D - 10 = 0 \qquad [1]$$

… and equating coefficients of sin(t)…

$$-6C + 7D = 0 \qquad [2]$$

[1] X 6/7: $$6C + \frac{36}{7}D - \frac{60}{7} = 0 \qquad [3]$$

[2] + [3]: $$7D + \frac{36}{7}D - \frac{60}{7} = 0 \quad \therefore \quad after\ some\ working,\ \boldsymbol{D = 60/85}$$

From [2]: $$-6C + 7\left(\frac{60}{85}\right) = 0 \quad \therefore \quad after\ some\ working,\ \mathbf{C = 14/17}$$

Now we bring everything together and finish…

$$GS = CF + PI$$

$$\boldsymbol{GS = Ae^{-2t} + Be^{-4t} + \frac{14}{17}cos(t) + \frac{60}{85}sin(t)}$$

That was hard going, but we solved it. These are laborious problems and much better suited to simulators (like MicroCap).