Pearson BTEC Levels 5 Higher Nationals in Engineering (RQF)

Unit 39: Further Mathematics

Unit Workbook 4

in a series of 4 for this unit

Learning Outcome 4

Ordinary Differential Equations

Contents

INTRODUCTION

Review models of engineering systems using ordinary differential equations

Differential equations:

Formation and solutions of first-order differential equations.

Applications of first-order differential equations e.g. RC and RL electric circuits, Newton's laws of cooling, charge and discharge of electrical capacitors and complex stresses and strains.

Formation and solutions of second-order differential equations.

Applications of second-order differential equations e.g. mass-spring-damper systems, information and energy control systems, heat transfer, automatic control systems and beam theory and RLC circuits.

Introduction to Laplace transforms for solving linear ordinary differential equations.

Applications involving Laplace transforms such as electric circuit theory, load frequency control, harmonic vibrations of beams, and engine governors.

4.1 First Order Differential Equations

Your solutions may be checked with this online calculator

4.1.1 Separation of Variables

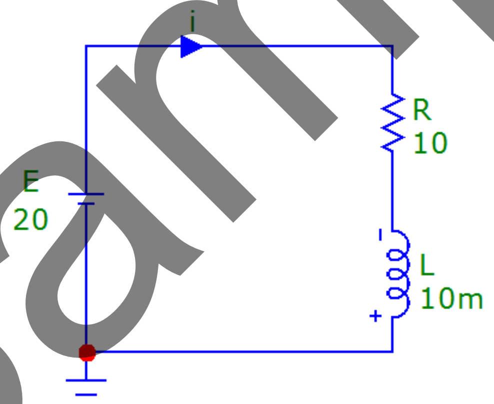
First Order differential equations involve a single derivative (such as $\frac{dy}{dx}$). Consider first order differential equations of the form…

$$\frac{dy}{dx} = f(x) \;\; or \;\; \frac{dy}{dx} = f(y) \;\; or \;\; \frac{dy}{dx} = f(x) \cdot f(y)$$

Each of these may be solved by direct integration after the separation of variables. By doing so we place the $y's$ on one side and the $x's$ on the other side. Series RL and series RC circuit analysis yields first order differential equations which may be treated in this way. Let's examine these circuits to see if we can first of all produce a differential equation for each of them, and then try to solve the RL circuit by separation of variables.

Worked Example 1

Analyse the RL circuit below and formulate a mathematical model which represents the circuit behaviour by a first order differential equation.



We need to make use of Kirchhoff's Voltage Law, which states that the voltages in a closed loop sum to zero. We note the voltages across the resistor and inductor as…

$$V_R = iR$$

$$V_L = L\frac{di}{dt}$$

So we may now write…

$$E = V_R + V_L$$

$$\therefore \quad E = iR + L\frac{di}{dt}$$

We always try to make the coefficient of the differential 1 so dividing through by L will achieve this…

$$\frac{E}{L} = \frac{iR}{L} + \frac{L}{L} \cdot \frac{di}{dt}$$

Tidying this up and re-arranging gives…

$$\frac{di}{dt} + \left(\frac{R}{L}\right) i = \frac{E}{L}$$

All we need to do now is to insert those component values…

$$\frac{di}{dt} + \left(\frac{10}{0.01}\right) i = \frac{20}{0.01}$$

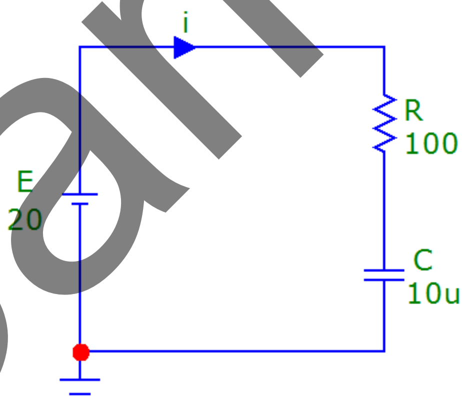
Which allows us now to write the final differential equation to represent this circuit…

$$\boldsymbol{\frac{di}{dt} + 1000i = 2000}$$

Let's now perform the same analysis on an RC circuit…

Worked Example 2

Analyse the RC circuit below and formulate a mathematical model which represents the circuit behaviour by a first order differential equation.



We again make use of Kirchhoff's Voltage Law, which states that the voltages in a closed loop sum to zero. We note the voltages across the resistor and capacitor as…

$$V_R = iR$$

$$V_C = \frac{1}{C}\int i\, dt$$

So we may now write…

$$E = V_R + V_C$$

$$\therefore \quad E = iR + \frac{1}{C}\int i\,dt$$

Multiply throughout by C…

$$EC = RCi + \int i\,dt$$

Then differentiate each term with respect to t…

$$0 = RC\frac{di}{dt} + i$$

Divide throughout by RC and tidy…

$$\frac{di}{dt} + \left(\frac{1}{RC}\right)i = 0$$

All we need to do now is to insert those component values…

$$\frac{di}{dt} + \left(\frac{1}{100 \times 10 \times 10^{-6}}\right)i = 0$$

Which allows us now to write the final differential equation to represent this circuit…

$$\boldsymbol{\frac{di}{dt} + 1000i = 0}$$

Worked Example 3

Produce an analytical solution for the first order differential equation developed in Worked Example 1 to find an expression for the current. Assume $i = 0$ when $t = 0$.

We had…

$$\frac{di}{dt} + 1000i = 2000$$

We start by transposing…

$$\frac{di}{dt} = 2000 - 1000i$$

Separate the variables (the variables are i and t)…

$$dt = \frac{di}{2000 - 1000i}$$

Integrate both sides with respect to t…