Pearson BTEC Levels 4 Higher Nationals in Engineering (RQF)

Unit 8: Mechanical Principles

Unit Workbook 4

in a series of 4 for this unit

Learning Outcome 4

Translational and Rotational Mass-Spring Systems

1 Types of motion:

1.1 Simple harmonic motion.

A particle is said to be under Simple Harmonic Motion (SHM) if its acceleration along a line is directly proportional to its displacement from a fixed point on that line.
Consider the motion of a particle A, rotating in a circle with a constant angular velocity ω, as shown in Figure 1 (a).

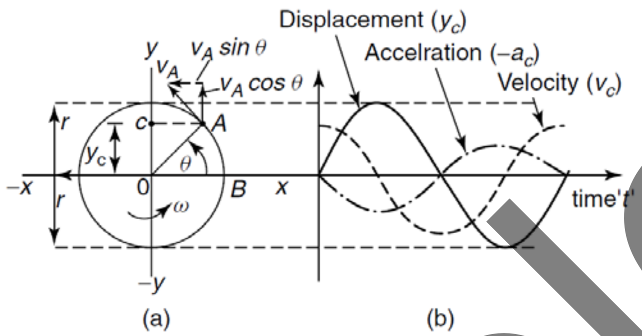


Figure 1 Simple Harmonic Motion

Consider now the vertical displacement of A from the x-axis, as shown by the distance y_C. If P is rotating at a constant angular velocity ω then the periodic time τ to travel an angular distance of 2π, is given by:

$$\boldsymbol{\tau = \frac{2\pi}{\omega}} \qquad \text{(Eq 1)}$$

Let f = frequency of motion C (in Hertz), where

$$\boldsymbol{f = \frac{1}{t} = \frac{\omega}{2\pi}} \qquad \text{(Eq 2)}$$

To determine whether SHM is taking place, consider the motion of A in the vertical direction (y-axis).
Now y_C = OA sin ωt, i.e.,

$\boldsymbol{y = r \sin \omega t,}$ where t = time in seconds (Eq 3)

Plotting of equation (Eq 3) against t results in the sinusoidal variation for displacement, as shown in Figure 1 (b).
We know that $\boldsymbol{v_A = \omega r}$, which is the tangential velocity of the particle A. From the velocity vector diagram, at the point A on the circle of Figure 1 (a),

$$\boldsymbol{v_c = v_A \cos\theta = v_A \cos \omega t} \qquad \text{(Eq 4)}$$

Plotting of equation (Eq 4) against t results in the sinusoidal variation for the velocity v_C, as shown in Figure 1 (b).
The centripetal acceleration of A $= a_A = \omega^2 r$

Now $a_C = -a_A \sin\theta,$ $\quad \therefore \boldsymbol{a_C = -\omega^2 r \sin \omega t}$ (Eq 5)

Plotting equation (Eq 5) against t results in the sinusoidal variation for the acceleration at C, a_C, as shown in Figure 1 (b).

Substituting equation (Eq 3) into equation (Eq 5) gives:

$$\boldsymbol{a_C = -\omega^2 y_C} \qquad \text{(Eq 6)}$$

Equation (Eq 6) shows that the acceleration along the y-axis is directly proportional to the displacement along this line, therefore the point C is moving with SHM. Now,

$T = \frac{2\pi}{\omega}$, but from equation (Eq 6) $\quad a_C = -\omega^2 y_C \quad$ i.e. $\quad \omega^2 = \frac{a}{y}$

Therefore, $\quad T = \frac{2\pi}{\sqrt{\frac{a}{y}}} \quad$ or $\quad T = 2\pi\sqrt{\frac{y}{a}} \quad$ i.e. $\quad T = 2\pi\sqrt{\frac{displacemet}{acceleration}}$

In general, from equation (Eq 6) $\quad \boldsymbol{a + \omega^2 y = 0} \quad$ (Eq 7)

1.2 Natural frequency of vibration in mass-spring systems.

1.2.1 The Spring-Mass System vibrating horizontally

Consider a mass m resting on a smooth surface and attached to a spring of stiffness k, as shown in Figure 2.

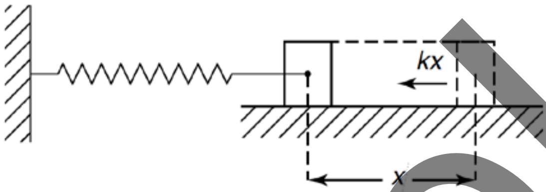


Figure 2 Horizontal Spring-Mass System

If the mass is given a small displacement x, the spring will exert a resisting force of kx,

i.e. $\quad F = -kx \quad$ But $\quad F = ma \quad$ Hence, $\quad ma = -kx$

or, $\quad ma + kx = 0 \quad$ or, $\quad \boldsymbol{a + \frac{k}{m}x = 0} \quad$ (Eq 8)

Equation (Eq 8) shows that this mass is oscillating (or vibrating) in SHM, or according to equation (Eq 7).

Comparing (Eq 7) with (Eq 8), we see that; $\quad \omega^2 = \frac{k}{m} \quad$ from which $\quad \omega = \sqrt{\frac{k}{m}}$

Now $\quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad$ and f = frequency of oscillation or vibration.

i.e. $\quad \boldsymbol{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \quad$ (Eq 9)

1.2.1 The Spring-Mass System vibrating horizontally

Consider a mass m, supported by a vertical spring of stiffness k, as shown in Figure 3. In this equilibrium position, the mass has an initial downward static deflection of y_o. If the mass is given an additional downward displacement of y and then released, it will vibrate vertically.

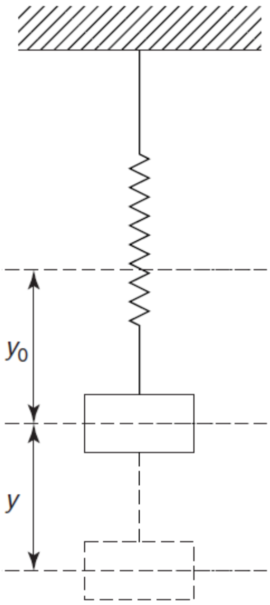


Figure 3 Vertical Spring-Mass System

The force exerted by the spring $= -k(y_0 + y)$

Therefore, $F = mg = -k(y_0 + y) = ma$ i.e. $f = mg = -ky_0 - ky = ma$

But $k_0 = mg$, hence, $F = mg - mg - ky = ma$

Thus, $ma + ky = 0$ or, $a + \frac{k}{m}y = 0$

i.e. SHM takes place and periodic time,

$$\boldsymbol{T = 2\pi\sqrt{\frac{m}{k}}} \qquad \text{(Eq 10)}$$

And frequency, $$\boldsymbol{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad \text{(Eq 11)}$$

Comparing equations (Eq 9) and (Eq 11), it can be seen that there is no difference in whether the spring is horizontal or vertical.

Equations (Eq 9) and (Eq 11) give the **natural frequency of vibration**, which tells us the frequency with which an object would vibrate when disturbed. For example, when you pluck a guitar string it vibrates with its natural frequencies (harmonics). The stiffness of the guitar string varies as you press it against a different fret and so does its frequency and the sound it ultimately produces.

All bodies have natural frequencies because all bodies have mass and stiffness, that is, there is repetitive inter-conversion between kinetic energy (stored by the mass) and Potential energy (stored by the stiffness). Mechanical vibration is essentially an interplay between inertial and elastic forces, so even air has natural frequencies dependent on the volume and shape of the enclosure (think acoustic cabinets or musical instruments!).

In the spring-mass system described here, once oscillations start they would carry on ad infinitum, unless some external force intervened. In real systems this external intervention is called damping and is essentially a leakage path for the energy in the system. ***Damping occurs in all real systems***.

2 Damped systems:

2.1 Restating the equations of motion.

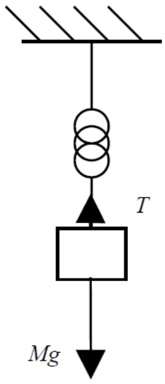


Figure 4 Undamped Mass-Spring System

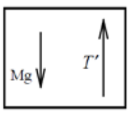
Xxx shows a mass, m suspended from a spring of natural length (l) and modulus of elasticity (λ). If the elastic limit of the spring is not exceeded and the mass hangs in equilibrium, the spring will extend by an amount, e, such that, by Hooke's law the tension in the spring, T, is given by;

$$\boldsymbol{T = \frac{\lambda e}{l}} \qquad \text{(Eq 12)}$$

For system equilibrium, this will be balanced by the weight so; $\boldsymbol{mg = T = \frac{\lambda e}{l}}$ (Eq 13)

If the spring is pulled down a further distance, x, (with x positive indicating downwards) the restoring force will now be the new tension in the spring, T′, given by $\mathrm{T}' = \frac{\lambda(e+x)}{l}$ and so the net force acting ***downwards*** is;

$$mg - \mathrm{T}' = \mathrm{mg} - \frac{\lambda(e+x)}{l} = mg - \frac{\lambda e}{l} - \frac{\lambda x}{l}$$



but from equation (Eq 13) $mg = \frac{\lambda e}{l}$

So, the net force acting ***downwards*** $= \frac{\lambda x}{l}$ (Eq 14)

From Newtons 2nd Law, $Force = mass\ x\ acceleration = m\frac{d^2x}{dt^2}$ (Eq 15)

So, combining equations (Eq 14) and (Eq 15) gives;

$$m\frac{d^2x}{dt^2} + \frac{\lambda x}{l} = 0 \qquad \text{(Eq 16)}$$

Equation (Eq 13) is a second-order differential equation with dependent variable x (displacement) and independent variable t (time) and system parameters m and k. TWO initial conditions are set, usually the mass's initial displacement from some datum and its initial velocity. (Note that velocity $= \frac{dx}{dt}$)

Since the system above is unforced, any motion of the mass will be due to the initial conditions ***ONLY***. Assume that the initial conditions are given as x (0) = -2 and v (0) = +4. Further, assume that ***downward*** is the positive direction, distance is in centimetres and t is in seconds. These initial conditions say that at t = 0, the mass is instantaneously 2cm ***above*** the datum and travelling with a velocity of 4 cm/s in the ***downwards*** direction.

2.2 The Unforced mass-spring-damper systems.

The above system is unrealistic since it does not account for damping. Damping can be introduced into the system physically, schematically and mathematically by incorporating all resistances into a dashpot as shown in Figure 5.

It can be shown that in such cases ***the resistance to motion is directly proportional to the velocity*** of the mass and, naturally, opposes the motion. This is not unreasonable since the faster the mass moves, the greater the resistance is exerted upon it (compare how much more difficult it is running, rather than walking, through water).

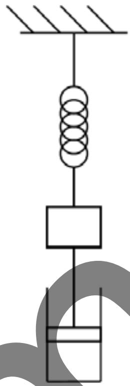


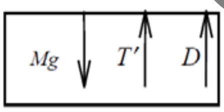
Figure 5 The Unforced Mass-Spring-Damper System

So, the damping force, D can be represented by $D = -R\frac{dx}{dt}$ (Eq 17)

And R is the constant of proportionality and is called the Damping Factor. For all real systems $R > 0$.

The inclusion of the damping factor modifies the equations of the previous case as follows:

Here, the net downward force will be;



$$mg - \mathrm{T}' - \mathrm{D} = \mathrm{mg} - \frac{\lambda(e+x)}{l} - R\frac{dx}{dt} = -\frac{\lambda x}{l} - R\frac{dx}{dt}$$

And, again using Newton's 2nd Law of motion, this results in;

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + \frac{\lambda x}{l} = 0 \qquad \text{(Eq 18)}$$

Or $$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = 0 \qquad \text{(Eq 19)}$$

Where $k = \frac{\lambda}{l}$